Online class # 07

Date: 01/08/2021

Chapter 05 (Vector Integration)

Time: 0930 – 1030

Video: https://youtu.be/CvpNijhqtOo

Vector Integration

Chapter 5

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int f \, d\vec{r}$$

$$\int \phi \, d\vec{s}$$

$$f(x, y, z)$$

$$\int \phi \, d$$

$$\int dx dy$$

$$\bar{r} = x i$$

$$\int dr = \int dx dy dz$$

$$r = x\hat{i} + y\hat{j}$$

$$dr = i dx + j dy$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\int f dr = \int f(i dx + j dy) = i \int f dx + j \int f dy$$

ORDINARY INTEGRALS OF VECTORS. Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u)$, $R_2(u)$, $R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$. If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u. The *definite integral* between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

This integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral*. If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In aerodynamics and fluid mechanics this integral is called the *circulation* of $\mathbf{A}$ about C, where $\mathbf{A}$ represents the velocity of a fluid.

THEOREM. If $\mathbf{A} = \nabla\phi$ everywhere in a region R of space, defined by $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$, $c_1 \leqq z \leqq c_2$, where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

1. $\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r}$ is independent of the path C in R joining P_1 and P_2.

2. $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ around any closed curve C in R.

In such case $\mathbf{A}$ is called a *conservative vector field* and ϕ is its *scalar potential.*

A vector field $\mathbf{A}$ is conservative if and only if $\nabla\times\mathbf{A} = \mathbf{0}$, or equivalently $\mathbf{A} = \nabla\phi$. In such case $\mathbf{A}\cdot d\mathbf{r} = A_1\,dx + A_2\,dy + A_3\,dz = d\phi$, an exact differential. See Problems 10-14.

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of the positive side of S is called a *positive* or *outward drawn* unit normal.

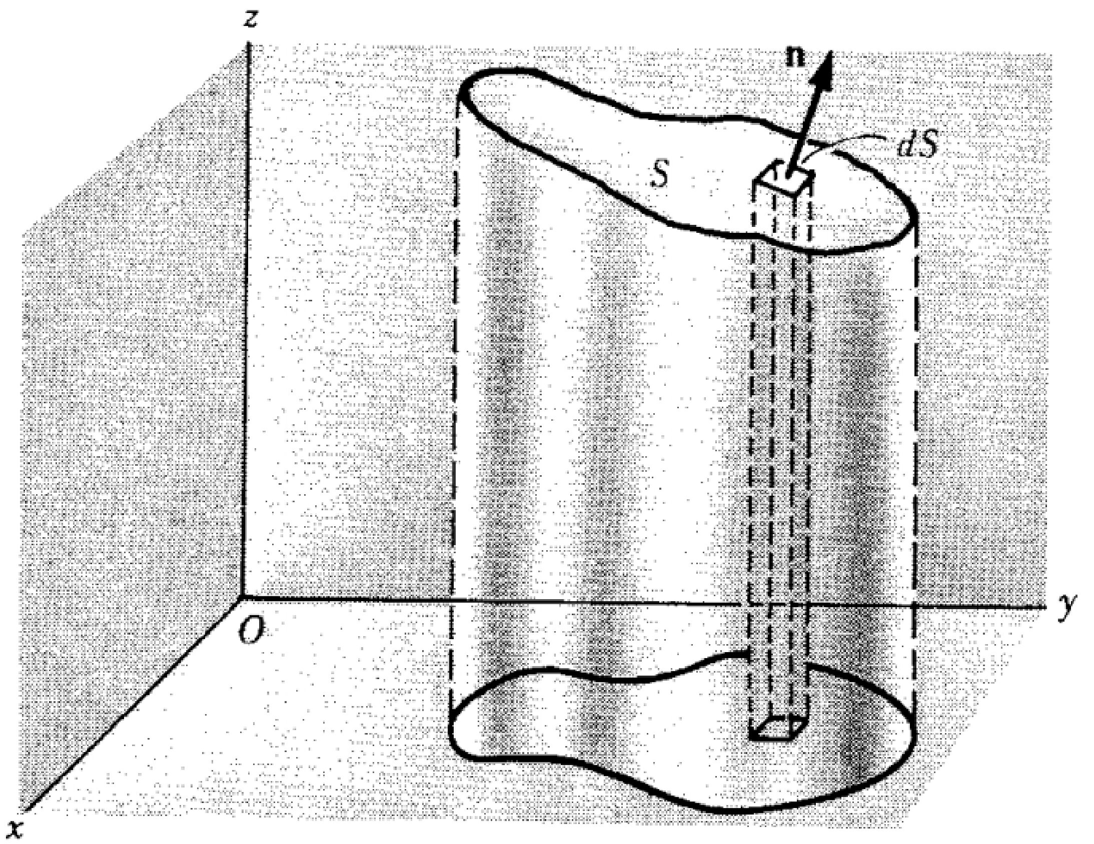
Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\, dS$. The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$

is an example of a surface integral called the *flux* of $\mathbf{A}$ over S. Other surface integrals are

$$\iint_S \phi\, dS, \quad \iint_S \phi\, \mathbf{n}\, dS, \quad \iint_S \mathbf{A}\times d\mathbf{S}$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus (see Problem 17).



VOLUME INTEGRALS. Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \mathbf{A}\, dV \quad \text{and} \quad \iiint_V \phi\, dV$$

are examples of *volume integrals* or *space integrals* as they are sometimes called. For evaluation of such integrals, see the Solved Problems.

1. If $\mathbf{R}(u) = (u - u^2)\,\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (a) $\int \mathbf{R}(u)\,du$ and (b) $\int_1^2 \mathbf{R}(u)\,du$.

$$\int R(u)\,du = \int \{(u-u^2)\,i + \quad {}^3 \quad -3k\}\,du$$

$$= i\int u\,du - i\int u^2\,du + 2j\int u^3\,du$$

1. If $\mathbf{R}(u) = (u - u^2)\,\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (a) $\int \mathbf{R}(u)\,du$ and (b) $\int_1^2 \mathbf{R}(u)\,du$.

$$\int_1^2 R(u)\,du = \left[\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\hat{i} + \frac{u^4}{2}\hat{j} - 3u\hat{k}\right]_1^2$$

$$= -\frac{5}{6}\hat{i} + \frac{15}{2}\hat{j} - 3\hat{k}$$

2. The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

$\Rightarrow \int \vec{a}\,dt = \vec{v} + ?$

$\boxed{\vec{v}(t=0) = 0}$

$\bar{a} = \frac{d\bar{v}}{dt}$

$\boxed{\vec{r}(t=0) = 0}$

$\int \bar{v}\,dt = \vec{r} + ?$

$\bar{v} = \frac{d\bar{r}}{dt}$

$\vec{v}(t) = \int \vec{a}\,dt = 12i \int \cos 2t\,dt - 8j \int \sin 2t\,dt + 16k \int t\,dt$

$\vec{v}(t) = \frac{12i}{2}\sin 2t + \frac{8j}{2}\cos 2t + \frac{16k}{2}t^2 + C$

$$v(t=0) = 0 + 4\hat{j} + 0 + c = 0$$

$$\boxed{c = -4\hat{j}}$$

$$\vec{v}(t) = \underline{\hat{i}\,6\sin 2t} + \hat{j}(4\cos 2t - 4) + \hat{k}\,8t^2$$

$$\vec{r} = \int \vec{v}(t)\,dt = -\frac{\hat{i}6}{2}\cos 2t + \hat{j}\left(\frac{4}{2}\sin 2t - 4t\right) + \hat{k}\,\frac{8}{3}t^3 + c'$$

$$\vec{r}(t=0) = -3\hat{i} + c' = 0$$

$$\Rightarrow c' = +3\hat{i}$$

$$\vec{r}(t) = (3 - 3\cos 2t)\,i + (2\sin 2t - 4t) + \frac{8}{3}t^3$$

3. Evaluate $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt$.

$$\frac{d}{dt}\left(\mathbf{A} \times \frac{d\mathbf{A}}{dt}\right) = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2} + \frac{d\mathbf{A}}{dt} \times \frac{d\mathbf{A}}{dt} = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}$$

Integrating, $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt = \int \frac{d}{dt}\left(\mathbf{A} \times \frac{d\mathbf{A}}{dt}\right) dt = \mathbf{A} \times \frac{d\mathbf{A}}{dt} + \mathbf{c}$.

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(a) $x = t$, $y = t^2$, $z = t^3$.

(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(c) the straight line joining (0,0,0) and (1,1,1).

x y z

(a)

$\vec{A} = (3t^2 + 6t^2)i - 14t^5 j + 20t^7 k$

$\vec{r} = xi + yj + zk$

$d\vec{r} = dx\,i + dy\,j + dz\,k$

$= i\,dt + j\,d(t^2) + k\,d(t^3)$

$= i\,dt + j\,2t\,dt + k\,3t^2\,dt$

$= i\,dt$

$d(t^2) = 2t\,dt$

$$A(x, y, z) = x^2 y + yz + z^2$$

$$x = t$$

$$y = t^2$$

$$z = t^3$$

$$A(t) = t^4 + t^5 + t^6$$

$\cancel{\int \bar{A} \cdot d\vec{r}}$

$$\frac{(x^2)^{7/2}}{7/2} = \frac{(x^2)^{5/2+1}}{\frac{5}{2}+1} = \int (x^2)^{5/2} d(x^2)$$

$$\frac{2x^7}{7}$$

$$\int x^5 \, d(x^2)$$

$$\frac{2x^{7}}{7} = 2\int x^5 \cdot x \, dx$$

$$= 2\int x^6 dx$$

$$\int_C \bar{A}\cdot d\bar{r} = \int \left(i\,9t^2 - j\,14t^5 + k\,20t^7\right)\cdot\left(i\,dt + j\,2t\,dt + k\,3t^2\,dt\right)$$

$$= \int_0^1 \left(9t^2 - 28t^6 + 60t^9\right)dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1 = 5$$