Online class # 08

Date: 03/08/2021

Chapter 05 (Vector Integration)

Time: 0930 – 1030

Video: https://youtu.be/1nX05sTVlN4

Vector Integration

Chapter 5

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(a) $x = t,\ y = t^2,\ z = t^3$.

(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(c) the straight line joining (0,0,0) and (1,1,1).

(b)

$$\int \vec{A}\cdot d\vec{r} = \int_0^1 (3x^2 + 6y)\,dx$$

$$= \int_0^1 3x^2\,dx = 1$$

$x = 0$, $x = 1$ | dx

$y = 0$

$dy = dz = 0$

$$\boxed{d\vec{r} = i\,dx}$$

$0 \to 1$

$$\int \vec{A}\cdot d\vec{r} = \int_0^1 (4y(0))\,dy = 0$$

$dx = dz = 0$

$x = 1$

$d\vec{r} = \hat{j}\,dy$

$y : 0 \to 1$

$x = 1$

$y = 1$

$dx = dy = 0$

$d\vec{r} = \hat{k}\,dz$

$z : 0 \to 1$

$$\int \vec{A}\cdot d\vec{r}_1 = \int_0^1 20\cdot 1\cdot z\,dz = 20/3$$

$$\int_C \vec{A}\cdot d\vec{r} = 1 + 0 + \frac{20}{3} = 23/3$$

$$\int \bar{A}\cdot d\vec{r} = \int (\quad)(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$t=0$

$(0,0,0) \to (1,1,1)$

$t=1$

$$= \int_0^1 (3x^2+6y)\,dx - \int_0^1 14yz\,dy + \int_0^1 20xz^2\,dz$$

$x = t$

$y = t$

$z = t$

$$\int \bar{A}\cdot d\vec{r} = \frac{13}{3}$$

$$d\vec{r} = \hat{i}\,dt + \hat{j}\,dt + \hat{k}\,dt$$

$$\vec{A} = \hat{i}(3t^2+6t) - \hat{j}\,14t^2 + \hat{k}\,20t^3$$

$t : 0 \to 1$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

$$\int_{t=1}^{2} \vec{F}\cdot d\vec{r} = 303$$

$d\bar{r} = i\,dx + j\,dy + k\,dz$

$x = t^2 + 1$

$dx = 2t\,dt$

$y = 2t^2$

$dy = 4t\,dt$

$dz = 3t^2\,dt$

$d\vec{r} = i\,2t\,dt + j\,4t\,dt + k\,3t^2\,dt$

$F = 3xy\,i - 5z\,j + 10x\,k$

$= 3(t^2+1)2t^2\,i - 5t^3\,j + 10(t^2+1)k$

$=$

8. If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y = 2x^2$, from $(0,0)$ to $(1,2)$.

$(1,2) \to (0,0)$ $\qquad dy = 4x\,dx$

Since the integration is performed in the xy plane ($z=0$), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$\int y\,dx \qquad y = 2x^2 \qquad 0 \to 1$ $\qquad x: 0 \to 1 \qquad 1 \to 0$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C 3xy\,dx - y^2\,dy$$

First Method. Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t,\ y = 2t^2$. Points $(0,0)$ and $(1,2)$ correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} 3(t)(2t^2)\,dt - (2t^2)^2\,d(2t^2) = \int_{t=0}^{1} (6t^3 - 16t^5)\,dt = -\frac{7}{6}$$

$7/6$

Second Method. Substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} 3x(2x^2)\,dx - (2x^2)^2\,d(2x^2) = \int_{x=0}^{1} (6x^3 - 16x^5)\,dx = -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from $(1,2)$ to $(0,0)$, the value of the integral would have been $7/6$ instead of $-7/6$.

9. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$

In the plane $z=0$, $\mathbf{F} = (2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ so that the work done is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(2x-y)\mathbf{i} + (x+y)\mathbf{j} + (3x-2y)\mathbf{k}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}]$$

$$= \int_C (2x-y)\,dx + (x+y)\,dy$$

$x = 3\cos t$

$y = 3\sin t$

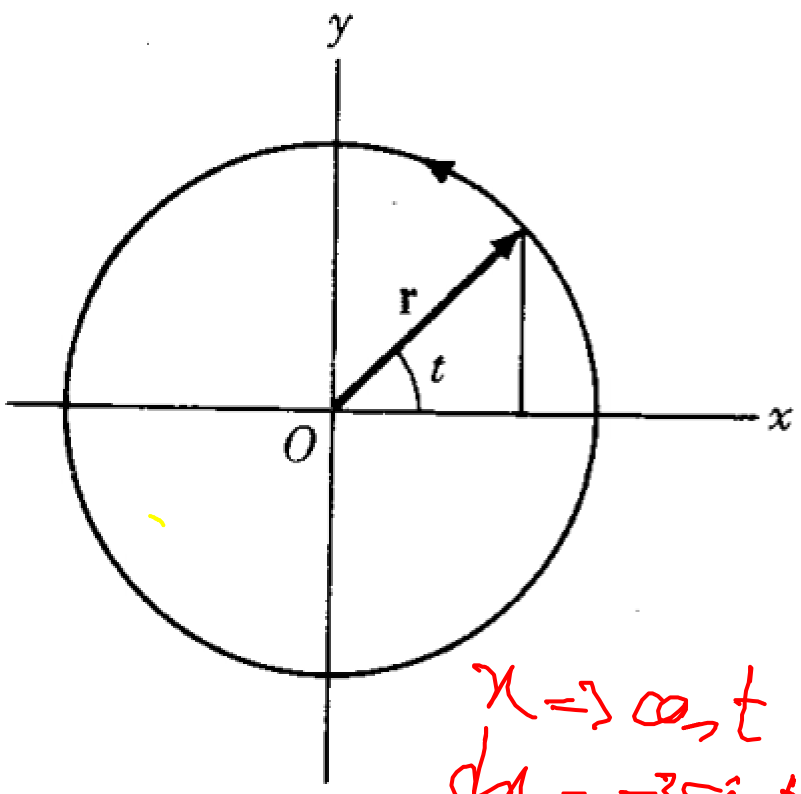
$$= -3\int_0^{2\pi} (6\cos t - 3\sin t)\sin t\,dt + 3\int_0^{2\pi} (3\cos t + 3\sin t)\cos t\,dt$$

$$\int \sin t \cos t \, dt =$$

$$= \int_0^{2\pi} \left[-18 \sin t \cos t + 9 \sin^2 t + 9 \cos^2 t + 9 \sin t \cos t \right] dt$$

$$= \int_0^{2\pi} \left[9 - 9 \sin t \cos t \right] dt$$

$$= 9 \left[t \right]_0^{2\pi} = 18\pi$$



$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$$
$$= 3 \cos t\, \mathbf{i} + 3 \sin t\, \mathbf{j}$$

$x = 3 \cos t$

$dx = -3 \sin t \, dt$

$dy = 3 \cos t \, dt$

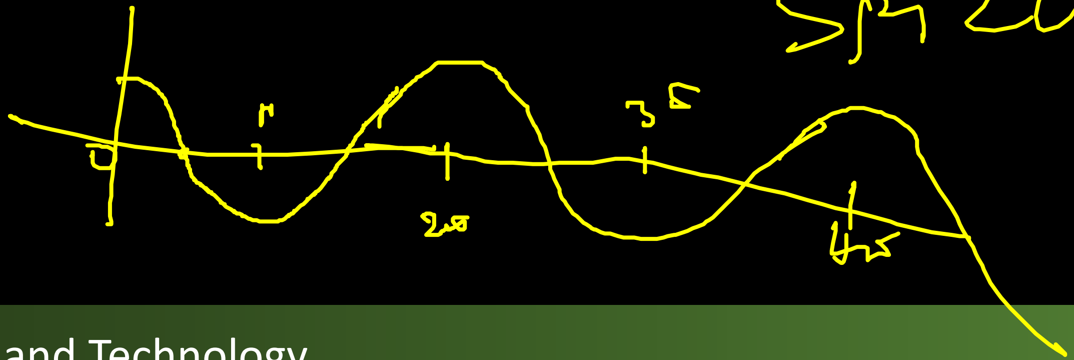
$$\int_0^{2\pi} \sin t \cos t \, dt = \frac{1}{2}\int_0^{2\pi} 2 \sin t \cos t \, dt$$

$$= \frac{1}{2}\int_0^{2\pi} \sin 2t \, dt$$

$$= -\frac{1}{2}\left[\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$-\frac{1}{4}\left[1-1\right] = 0$$



$$\int_0^{2\pi} \sin t \cos t \, dt$$

$$= \int p \, dp$$

$$= \frac{p^2}{2}$$

$$= \left[\frac{\sin^2 t}{2}\right]_{t=0}^{2\pi}$$

$$= \frac{1}{2}\left[0 - 0\right] = 0$$

~~cos t~~

$$\sin t = p$$

$$\cos t \, dt = dp$$