

PHY 3101

Electrical Engineering Materials

Elementary Quantum Physics

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Commutator

$$[\hat{x}, \hat{p}] = i\hbar$$

Commutator of two operators

The commutator of two operators \hat{A} and \hat{B} is defined by

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A}, \hat{B}]\phi \equiv \hat{A}\hat{B}\phi - \hat{B}\hat{A}\phi$$

Why is it important?

Commutator of two operators

$$\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$$

$$\hat{B}\hat{A}\phi \equiv \hat{B}(\hat{A}\phi)$$

$\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ may not be the same

Commutator of two operators

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

Two operators are said to commute.

$$[\hat{A}, \hat{A}] = 0$$

Anticommutator

The anticommutator $\{\hat{A}, \hat{B}\}$ is defined by

$$\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$$

Position operator

The **position operator** \hat{x} that acting on functions of x gives another function of x as follows:

$$\hat{x} f(x) \equiv x f(x)$$

$$\hat{x}^k f(x) \equiv x^k f(x)$$

Momentum operator

The momentum operator acts on wave functions, which are functions of space and time to give another function of space and time:

$$\hat{p}\Psi(x, t) = p\Psi(x, t)$$

$$\hat{p} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} = -i\hbar \frac{\partial}{\partial x}$$

Momentum operator

Let us consider the wavefunction of a free particle:

$$\Psi(x, t) = e^{i(xp - Et)/\hbar}$$

$$\begin{aligned}\hat{p}\Psi(x, t) &= -i\hbar \frac{\partial}{\partial x} e^{i(xp - Et)/\hbar} \\ &= (-i\hbar) \left(\frac{ip}{\hbar} \right) e^{i(xp - Et)/\hbar} \\ &= p\Psi(x, t)\end{aligned}$$

Linear operator

$$\hat{A}(a\phi) \equiv a\hat{A}\phi$$

$$\hat{A}(\phi_1 + \phi_2) \equiv \hat{A}\phi_1 + \hat{A}\phi_2$$

$$(\hat{A} + \hat{B})\phi \equiv \hat{A}\phi + \hat{B}\phi$$

$$\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{x} = x$$

$$\hat{p} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} = -i\hbar \frac{\partial}{\partial x}$$

We have operators \hat{x} and \hat{p} that are clearly somewhat related. We would like to know their commutator $[\hat{x}, \hat{p}]$. For this we let $[\hat{x}, \hat{p}]$ act on some arbitrary function $\phi(x)$ and then attempt simplification.

Commutator $[\hat{x}, \hat{p}]$

$$\begin{aligned} [\hat{x}, \hat{p}] \phi(x) &= (\hat{x} \hat{p} - \hat{p} \hat{x}) \phi(x) \\ &= \hat{x} \hat{p} \phi(x) - \hat{p} \hat{x} \phi(x) \\ &= \hat{x} (\hat{p} \phi(x)) - \hat{p} (\hat{x} \phi(x)) \\ &= \hat{x} \left(-i\hbar \frac{\partial \phi(x)}{\partial x} \right) - \hat{p} (x \phi(x)) \end{aligned}$$

Commutator $[\hat{x}, \hat{p}]$

$$\begin{aligned} [\hat{x}, \hat{p}]\phi(x) &= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \frac{\partial}{\partial x} (x\phi(x)) \\ &= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \phi(x) \\ &= i\hbar \phi(x) \end{aligned}$$

Commutator $[\hat{x}, \hat{p}]$

$$[\hat{x}, \hat{p}] \phi(x) = i\hbar \phi(x)$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$[\hat{p}, \hat{x}] = -[\hat{x}, \hat{p}] = -i\hbar$$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{\mathbf{r}} \equiv (\hat{x}, \hat{y}, \hat{z})$$

$$\hat{\mathbf{p}} \equiv -i\hbar\nabla$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Commutator $[\hat{x}, \hat{p}]$

operator	\hat{p}_x	\hat{p}_y	\hat{p}_z
\hat{x}	$\pm i\hbar$	0	0
\hat{y}	0	$\pm i\hbar$	0
\hat{z}	0	0	$\pm i\hbar$

$$[\hat{x}, \hat{p}_x] = i\hbar$$

$$[\hat{p}_x, \hat{x}] = -i\hbar$$

Commutator $[\hat{x}, \hat{p}]$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

The Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{p}_k = -i\hbar \frac{\partial}{\partial x_k}, \quad k = 1, 2, 3.$$

$$(x_1, x_2, x_3) = (x, y, z)$$

$$(p_1, p_2, p_3) = (p_x, p_y, p_z)$$

Operators

operator	position	momentum
1	\hat{x}	\hat{p}_x
2	\hat{y}	\hat{p}_y
3	\hat{z}	\hat{p}_z