# Quantum Mechanics I

## PHY 3103

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# Simple Harmonic Oscillator

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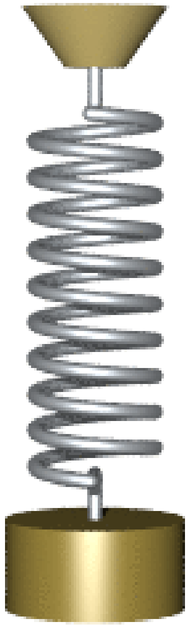
# Simple Harmonic Oscillator

It is one of those few problems that are important to all branches of physics. The harmonic oscillator provides a useful model for a variety of vibrational phenomena that are encountered, for instance, in classical mechanics, electrodynamics, statistical mechanics, solid state, atomic, nuclear, and particle physics. In quantum mechanics, it serves as an invaluable tool to illustrate the basic concepts and the formalism.



# The Hamiltonian of SHO

The total energy $$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Here $\omega = \sqrt{k/m}$ is angular frequency of the oscillation.

The Hamiltonian $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

$$\boxed{V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2.}$$

# Factorizing the Hamiltonian

$$\boxed{\hat{H} = \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)}$$

$$a^2 + b^2 = (a - ib)(a + ib)$$

$$\begin{aligned}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) \\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[\hat{x},\, \hat{p}] \\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\end{aligned}$$

# Factorizing the Hamiltonian

We now define $\hat{A} = \hat{x} + \dfrac{i\hat{p}}{m\omega}$

and its Hermitian conjugate $\hat{A}^\dagger = \hat{x} - \dfrac{i\hat{p}}{m\omega}$

We therefore have $\hat{x}^2 + \dfrac{\hat{p}^2}{m^2\omega^2} = \hat{A}^\dagger \hat{A} + \dfrac{\hbar}{m\omega}$

Therefore, the Hamiltonian $\boxed{\hat{H} = \dfrac{1}{2} m\omega^2 \hat{A}^\dagger \hat{A} + \dfrac{1}{2}\hbar\omega.}$

# The commutator of $\hat{A}$ and $\hat{A}^\dagger$

$$[\hat{A},\ \hat{A}^\dagger] = \left[\hat{x} + \frac{i\hat{p}}{m\omega},\ \hat{x} - \frac{i\hat{p}}{m\omega}\right]$$

$$= -\frac{i}{m\omega}[\hat{x},\ \hat{p}] + \frac{i}{m\omega}[\hat{p},\ \hat{x}]$$

$$= \frac{2\hbar}{m\omega}$$

$$\boxed{\left[\sqrt{\frac{m\omega}{2\hbar}}\hat{A},\ \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^\dagger\right] = 1}$$

# Creation and annihilation operators

Annihilation operator $$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A},$$

Creation operator $$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^\dagger.$$

$$\boxed{[\hat{a},\ \hat{a}^\dagger] = 1}$$

# Creation and annihilation operators

Annihilation operator $$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A},$$

Creation operator $$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^\dagger.$$

$$\boxed{[\hat{a},\, \hat{a}^\dagger] = 1}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\hat{p} = \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a} - \hat{a}^\dagger\right)$$

# Factorized Hamiltonian of SHO

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

$$= \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$\boxed{E = \hbar\omega\left(N + \frac{1}{2}\right).}$$

$$\boxed{\hat{N} \equiv \hat{a}^{\dagger}\hat{a}.}$$

Number operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

# The ground state

On any normalized state $\psi$

$$\langle \hat{H} \rangle_\psi = (\psi, \hat{H}\psi) = \hbar\omega(\psi, \hat{a}^\dagger \hat{a}\psi) + \frac{1}{2}\hbar\omega(\psi, \psi)$$

$$= \hbar\omega(\hat{a}\psi, \hat{a}\psi) + \frac{1}{2}\hbar\omega \geq \frac{1}{2}\hbar\omega$$

$$\boxed{(\psi, \psi) \geq 0}$$

$$\boxed{\hat{H}\psi = E\psi}$$

$$\boxed{E \geq \frac{1}{2}\hbar\omega}$$

# The ground state wave function

$$\hat{a}\psi_0 = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\psi_0(x) = 0$$

$$\left(\hat{x} + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_0(x) = 0$$

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}x} = -\frac{m\omega}{\hbar}x\psi_0$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}}$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

# Operator manipulation

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\boxed{\hat{N} \equiv \hat{a}^\dagger\hat{a}.}$$

$$\boxed{\hat{a}\psi_0 = 0}$$

$$\boxed{\hat{N}\psi_0 = \hat{a}^\dagger\hat{a}\psi_0 = 0.}$$

Thus $\psi_0$ is an eigenstate of the operator $\hat{N}$ with an eigenvalue $N = 0$. Therefore $\psi_0$ is an energy eigenstate with energy $E_0$ given by

$$\boxed{E_0 = \hbar\omega\left(0 + \frac{1}{2}\right) = \frac{1}{2}\hbar\omega}$$

# Operator manipulation

$$[\hat{N},\ \hat{a}] = [\hat{a}^\dagger\hat{a},\ \hat{a}] = \hat{a}^\dagger[\hat{a},\ \hat{a}] + [\hat{a}^\dagger,\ \hat{a}]\hat{a} = -\hat{a}$$

$$[\hat{N},\ \hat{a}^\dagger] = [\hat{a}^\dagger\hat{a},\ \hat{a}^\dagger] = \hat{a}^\dagger[\hat{a},\ \hat{a}^\dagger] + [\hat{a}^\dagger,\ \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger$$

Similarly we can show

$$[\hat{N},\ \hat{a}^2] = [\hat{N},\ \hat{a}\hat{a}] = [\hat{N},\ \hat{a}]\hat{a} + \hat{a}[\hat{N},\ \hat{a}] = -\hat{a}\hat{a} + \hat{a}(-\hat{a}) = -2\hat{a}^2,$$

$$[\hat{N},\ (\hat{a}^\dagger)^2] = [\hat{N},\ \hat{a}^\dagger\hat{a}^\dagger] = [\hat{N},\ \hat{a}^\dagger]\hat{a}^\dagger + \hat{a}^\dagger[\hat{N},\ \hat{a}^\dagger] = \hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a}^\dagger = 2(\hat{a}^\dagger)^2$$

# Operator manipulation

$$[\hat{N},\, (\hat{a})^k] = -k(\hat{a})^k$$

$$[\hat{N},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^k$$

$$[\hat{a}^\dagger,\, (\hat{a})^k] = -k(\hat{a})^{k-1}$$

$$[\hat{a},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^{k-1}$$

$$\text{If } \hat{A}\psi = 0,\ \text{ then } \hat{A}\hat{B}\psi = [\hat{A},\, \hat{B}]\psi$$

# Fist excited states

Since $\hat{a}$ annihilates $\psi_0$ consider acting on the ground state with $\hat{a}^\dagger$. It is clear that $\hat{a}^\dagger$ cannot also annihilate $\psi_0$. If that would happen then acting with both sides of the commutator identity $[\hat{a},\, \hat{a}^\dagger] = 1$ on $\psi_0$ would lead to a contradiction: the left-hand side would vanish but the right-hand side would not. Thus consider the wave function

$$\boxed{\psi_1 \equiv \hat{a}^\dagger \psi_0}$$

We are going to show that this is an energy eigenstate.

# Fist excited states

$$\hat{N}\psi_1 = \hat{N}\hat{a}^\dagger\psi_0 = [\hat{N},\, \hat{a}^\dagger]\psi_0 = \hat{a}^\dagger\psi_0 = \psi_1$$

- $\psi_0$ is an eigenstate of the operator $\hat{N}$ with eigenvalue $N = 0$.
- $\psi_1$ is an eigenstate of the operator $\hat{N}$ with eigenvalue $N = 1$.
- $\hat{a}^\dagger$ acting on $\psi_0$ increases the eigenvalue of $\hat{N}$ by one unit.
- $\hat{a}^\dagger$ is called the *creation* operator or the *raising* operator.

# Fist excited states

Moreover

$$(\psi_1,\, \psi_1) = (\hat{a}^\dagger\psi_0,\, \hat{a}^\dagger\psi_0) = (\psi_0,\, \hat{a}\hat{a}^\dagger\psi_0)$$

$$= (\psi_0,\, [\hat{a},\, \hat{a}^\dagger]\psi_0) = (\psi_0,\, \psi_0) = 1$$

$\psi_1$ is normalized and is the wave function of first exited state.

Energy of the first exited state $$E_1 = \hbar\omega\left(1 + \frac{1}{2}\right) = \frac{3}{2}\hbar\omega$$

# Second Excited states

Next consider the state $\psi_2' \equiv \hat{a}^\dagger \hat{a}^\dagger \psi_0.$

$$\boxed{\hat{N}\psi_2' = \hat{N}\hat{a}^\dagger\hat{a}^\dagger\psi_0 = [\hat{N},\, \hat{a}^\dagger\hat{a}^\dagger]\psi_0 = 2\hat{a}^\dagger\hat{a}^\dagger\psi_0 = 2\psi_2'}$$

$$\boxed{\psi_2' \text{ is a state with number } N = 2 \text{ and energy } E_2 = \tfrac{5}{2}\hbar\omega.}$$

$$\begin{aligned}(\psi_2',\, \psi_2') &= (\hat{a}^\dagger\hat{a}^\dagger\psi_0,\, \hat{a}^\dagger\hat{a}^\dagger\psi_0) = (\psi_0,\, \hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\psi_0) = (\psi_0,\, \hat{a}[\hat{a},\, \hat{a}^\dagger\hat{a}^\dagger]\psi_0) \\ &= (\psi_0,\, 2\hat{a}\hat{a}^\dagger\psi_0) = 2(\psi_0,\, \psi_0) = 2.\end{aligned}$$

# Excited state wave functions

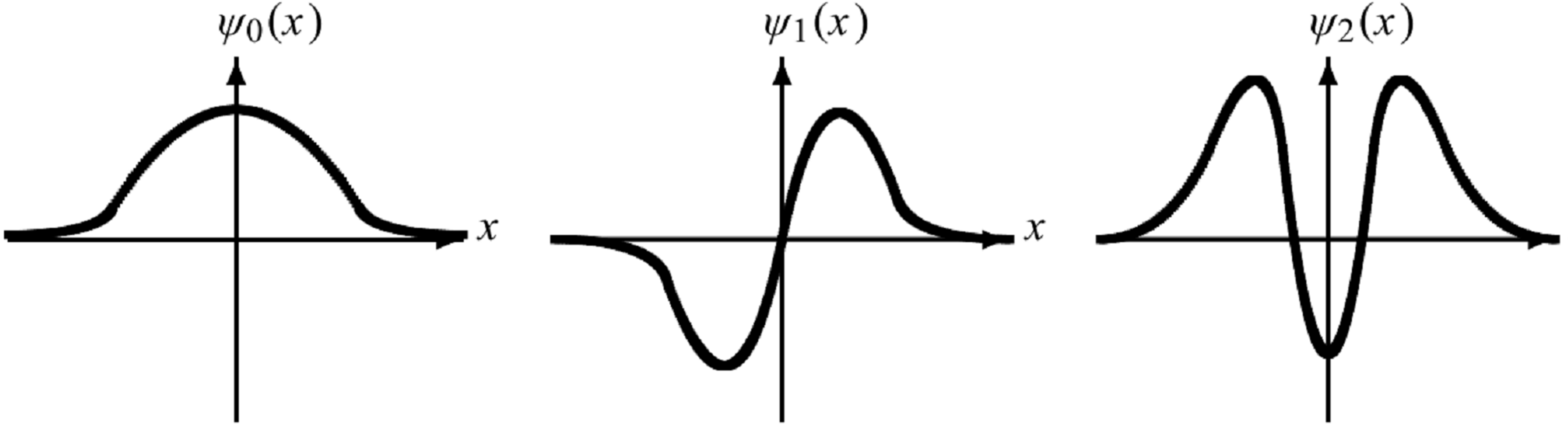
The properly normalized wave function is

$$\boxed{\psi_2 \equiv \frac{1}{\sqrt{2}}\hat{a}^\dagger\hat{a}^\dagger\psi_0.}$$

$$\boxed{\psi_3 \equiv \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\psi_0 = \frac{1}{\sqrt{3!}}(\hat{a}^\dagger)^3\psi_0}$$

$$\boxed{\psi_n \equiv \frac{1}{\sqrt{n!}}\underbrace{\hat{a}^\dagger\cdots\hat{a}^\dagger}\psi_0 = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0.}$$

# Shapes of the first three wave functions of the SHO




$$\boxed{\psi_n \equiv \frac{1}{\sqrt{n!}} \underbrace{\hat{a}^\dagger \cdots \hat{a}^\dagger}\, \psi_0 = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n \psi_0.}$$

# Eigenvalue of $\psi_n$ for the operators $\hat{N}$ and $\hat{H}$

$$
\begin{aligned}
\hat{N}\psi_n &= \frac{1}{\sqrt{n!}}\hat{N}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}\hat{a}^\dagger\hat{a}\,(\hat{a}^\dagger)^n\psi_0 \\
&= \frac{1}{\sqrt{n!}}\hat{a}^\dagger[\hat{a},\,(\hat{a}^\dagger)^n]\psi_0 = \frac{1}{\sqrt{n!}}\hat{a}^\dagger\,n(\hat{a}^\dagger)^{n-1}\psi_0 \\
&= \frac{n}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0 = n\,\psi_n
\end{aligned}
$$

Since for the operator $\hat{N}$ the eigenvalue of $\psi_n$ is $n$, the energy eigenvalue $E_n$ is given be

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$$

# Properties of of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{a}\psi_n = \hat{a}\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}[\hat{a},\ (\hat{a}^\dagger)^n]\psi_0 = \frac{n}{\sqrt{n!}}(\hat{a}^\dagger)^{n-1}\psi_0$$

$$= \frac{n}{\sqrt{n!}}\sqrt{(n-1)!}\,\psi_{n-1} = \sqrt{n}\,\psi_{n-1}$$

$$\begin{aligned} \hat{a}^\dagger\psi_n &= \hat{a}^\dagger\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^{n+1}\psi_0 \\ &= \frac{1}{\sqrt{n!}}\sqrt{(n+1)!}\,\psi_{n+1} = \sqrt{(n+1)}\,\psi_{n+1} \end{aligned}$$

# Orthonormality of eigenstates

Eigenstates are orthonormal: $\boxed{(\psi_m, \psi_n) = \delta_{mn}}$

Kronecker delta

$$\boxed{\delta_{mn} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}}$$

$$\boxed{\begin{aligned} \hat{a}\psi_n &= \sqrt{n}\,\psi_{n-1}, \\ \hat{a}^\dagger\psi_n &= \sqrt{(n+1)}\,\psi_{n+1}. \end{aligned}}$$

# Expectation values of operators

$$
\begin{aligned}
\langle \hat{x} \rangle_{\psi_n} &= (\psi_n,\, \hat{x}\psi_n) = \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, (\hat{a}+\hat{a}^\dagger)\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}\psi_n + \hat{a}^\dagger\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}\psi_n) + \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}^\dagger\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \sqrt{n}\,\psi_{n-1}) + \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \sqrt{(n+1)}\,\psi_{n+1}) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,\sqrt{n}(\psi_n,\, \psi_{n-1}) + \sqrt{\frac{\hbar}{2m\omega}}\,\sqrt{(n+1)}(\psi_n,\, \psi_{n+1}) \\
&= 0.
\end{aligned}
$$

# Expectation values of operators

$$
\begin{aligned}
\langle \hat{x}^2 \rangle_{\psi_n} &= (\psi_n,\, \hat{x}^2 \psi_n) = \frac{\hbar}{2m\omega}(\psi_n,\, (\hat{a} + \hat{a}^\dagger)^2 \psi_n) \\
&= \frac{\hbar}{2m\omega}(\psi_n,\, (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger)\psi_n) \\
&= \frac{\hbar}{2m\omega}(\psi_n,\, (\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger)\psi_n) \\
&= \frac{\hbar}{2m\omega}(\psi_n,\, (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})\psi_n) \\
&= \frac{\hbar}{2m\omega}(\psi_n,\, (1 + 2\hat{N})\psi_n) \\
&= \frac{\hbar}{2m\omega}(1 + 2n)
\end{aligned}
$$

$$\boxed{\hat{a}\hat{a}^\dagger = [\hat{a},\, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a} = 1 + \hat{N}}$$

# Expectation values of operators

$$\boxed{\langle \hat{p} \rangle_{\psi_n} = 0}$$

$$\boxed{\langle \hat{p}^2 \rangle_{\psi_n} = \frac{m\hbar\omega}{2}(1 + 2n).}$$

$$\boxed{\langle \hat{x} \rangle_{\psi_n} = 0}$$

$$\boxed{\langle \hat{x}^2 \rangle_{\psi_n} = \frac{\hbar}{2m\omega}(1 + 2n)}$$

$$\boxed{\frac{m\omega^2}{2} \langle \hat{x}^2 \rangle_{\psi_n} = \frac{1}{2m} \langle \hat{p}^2 \rangle_{\psi_n} = \frac{1}{2} \langle \hat{H} \rangle_{\psi_n}}$$

# Uncertainty principle

$$\boxed{\langle \hat{x} \rangle_{\psi_n} = 0}$$

$$\boxed{\langle \hat{x}^2 \rangle_{\psi_n} = \frac{\hbar}{2m\omega}(1 + 2n)}$$

$$\boxed{\langle \hat{p} \rangle_{\psi_n} = 0}$$

$$\boxed{\langle \hat{p}^2 \rangle_{\psi_n} = \frac{m\hbar\omega}{2}(1 + 2n).}$$

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$$

$$\Delta x \Delta p = \left( n + \frac{1}{2} \right) \hbar$$

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$

# Harmonic oscillator in 3D

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,y,z)\right]\psi(x,y,z) = E\psi(x,y,z)$$

$$\boxed{\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

$$\boxed{V(x,y,z) = V_x(x) + V_y(y) + V_z(z)}$$

$$\left[\hat{H}_x + \hat{H}_y + \hat{H}_z\right]\psi(x,y,z) = E\psi(x,y,z)$$

# Harmonic oscillator in 3D

$$\hat{H}_x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_x(x)$$

$$\hat{H}_y = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + V_y(y)$$

$$\hat{H}_z = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V_z(z)$$

$$\boxed{\psi(x, y, z) = X(x)Y(y)Z(z)}$$

# Harmonic oscillator in 3D

$$\left[-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}+V_x(x)\right]+\left[-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}+V_y(y)\right]+\left[-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2}+V_z(z)\right]=E$$

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_x(x)\right]X(x)=E_x X(x)$$

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+V_y(y)\right]Y(y)=E_y Y(y)$$

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2}+V_z(z)\right]Z(z)=E_z Z(z)$$

$$E_x+E_y+E_z=E$$

# Anisotropic harmonic oscillator

$$V(x,y,z) = \frac{1}{2}m\,\omega_x^2\,\hat{x}^2 + \frac{1}{2}m\,\omega_x^2\,\hat{y}^2 + \frac{1}{2}m\,\omega_x^2\,\hat{z}^2$$

$$\boxed{E_{n_x\,n_y\,n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \left(n_x + \frac{1}{2}\right)\hbar\omega_x + \left(n_y + \frac{1}{2}\right)\hbar\omega_y + \left(n_z + \frac{1}{2}\right)\hbar\omega_z}$$

$$\boxed{\psi_{n_x\,n_y\,n_z}(x,y,z) = X_{n_x}(x)\,Y_{n_y}(y)\,Z_{n_z}(z)}$$

# Isotropic harmonic oscillator

$$\boxed{\omega_x = \omega_y = \omega_z = \omega}$$

$$\boxed{E_{n_x\,n_y\,n_z} = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega}$$

| $n$ | $2E_n/(\hbar\omega)$ | $(n_x n_y n_z)$ | $g_n$ |
|---|---|---|---|
| 0 | 3 | (000) | 1 |
| 1 | 5 | (100), (010), (001) | 3 |
| 1 | 7 | (200), (020), (002) | 6 |
| | | (110), (101), (001) | |
| 3 | 9 | (300), (030), (003) | 10 |
| | | (210), (201), (021) | |
| | | (120), (102), (012) | |
| | | (111) | |