EEE 2105
Electrical Engineering Materials

Elementary Quantum Physics

Dr. Mohammad Abdur Rashid

Commutator

$$\boxed{[\hat{x},\, \hat{p}] = i\hbar}$$

Commutator of two operators

The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined by

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A}, \hat{B}]\phi \equiv \hat{A}\hat{B}\phi - \hat{B}\hat{A}\phi$$

Why is it important?

Commutator of two operators

$$\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$$

$$\hat{B}\hat{A}\phi \equiv \hat{B}(\hat{A}\phi)$$

$\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ may not be the same

Commutator of two operators

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

Two operators are said to commute.

$$[\hat{A}, \hat{A}] = 0$$

Anticommutator

The anticommutator $\{\hat{A}, \hat{B}\}$ is defined by

$$\boxed{\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}}$$

Position operator

The **position operator** $\hat{x}$ that acting on functions of x gives another function of x as follows:

$$\hat{x}f(x) \equiv xf(x)$$

$$\hat{x}^k f(x) \equiv x^k f(x)$$

Momentum operator

The momentum operator acts on wave functions, which are functions of space and time to give another function of space and time:

$$\hat{p}\Psi(x,t) = p\Psi(x,t)$$

$$\boxed{\hat{p} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}}$$

Momentum operator

Let us consider the wavefunction of a free particle:

$$\Psi(x,t) = e^{i(xp-Et)/\hbar}$$

$$\begin{aligned}\hat{p}\Psi(x,t) &= -i\hbar\frac{\partial}{\partial x}e^{i(xp-Et)/\hbar} \\ &= (-i\hbar)\left(\frac{ip}{\hbar}\right)e^{i(xp-Et)/\hbar} \\ &= p\Psi(x,t)\end{aligned}$$

Linear operator

$$\hat{A}(a\phi) \equiv a\hat{A}\phi$$

$$\hat{A}(\phi_1 + \phi_2) \equiv \hat{A}\phi_1 + \hat{A}\phi_2$$

$$(\hat{A} + \hat{B})\phi \equiv \hat{A}\phi + \hat{B}\phi$$

$$\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{x} = x$$

$$\hat{p} \equiv \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}$$

We have operators $\hat{x}$ and $\hat{p}$ that are clearly somewhat related. We would like to know their commutator $[\hat{x}, \hat{p}]$. For this we let $[\hat{x}, \hat{p}]$ act on some arbitrary function $\phi(x)$ and then attempt simplification.

Commutator $[\hat{x}, \hat{p}]$

$$
\begin{aligned}
[\hat{x},\, \hat{p}]\phi(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x})\phi(x) \\
&= \hat{x}\hat{p}\phi(x) - \hat{p}\hat{x}\phi(x) \\
&= \hat{x}(\hat{p}\phi(x)) - \hat{p}(\hat{x}\phi(x)) \\
&= \hat{x}\left(-i\hbar\frac{\partial\phi(x)}{\partial x}\right) - \hat{p}(x\phi(x))
\end{aligned}
$$

Commutator $[\hat{x}, \hat{p}]$

$$
\begin{aligned}
[\hat{x},\, \hat{p}]\phi(x) &= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \frac{\partial}{\partial x}(x\phi(x)) \\
&= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \phi(x) \\
&= i\hbar\, \phi(x)
\end{aligned}
$$

Commutator $[\hat{x}, \hat{p}]$

$$[\hat{x},\, \hat{p}]\phi(x) = i\hbar\, \phi(x)$$

$$\boxed{[\hat{x},\, \hat{p}] = i\hbar}$$

$$\boxed{[\hat{p},\, \hat{x}] = -[\hat{x},\, \hat{p}] = -i\hbar}$$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{\mathbf{p}} \equiv -i\hbar\nabla$$

$$\hat{\mathbf{r}} \equiv (\hat{x}, \hat{y}, \hat{z})$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Commutator $[\hat{x}, \hat{p}]$

operator	$\hat{p}_x$	$\hat{p}_y$	$\hat{p}_z$
$\hat{x}$	$\pm i\hbar$	0	0
$\hat{y}$	0	$\pm i\hbar$	0
$\hat{z}$	0	0	$\pm i\hbar$

$$[\hat{x},\, \hat{p}_x] = i\hbar$$

$$[\hat{p}_x,\, \hat{x}] = -i\hbar$$

Commutator $[\hat{x}, \hat{p}]$

$$\boxed{[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}}$$

The Kronecker delta: $$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Commutator $[\hat{x}, \hat{p}]$

$$\hat{p}_k = -i\hbar \frac{\partial}{\partial x_k}, \qquad k = 1, 2, 3.$$

$$(x_1, x_2, x_3) = (x, y, z)$$

$$(p_1, p_2, p_3) = (p_x, p_y, p_z)$$

Operators

operator	position	momentum
1	$\hat{x}$	$\hat{p}_x$
2	$\hat{y}$	$\hat{p}_y$
3	$\hat{z}$	$\hat{p}_z$