EEE 2105 Electrical Engineering Materials

Elementary Quantum Physics

Dr. Mohammad Abdur Rashid



Jashore University of Science and Technology

Dr Rashid, 2024

Commutator

$$[\hat{x},\,\hat{p}]=i\hbar$$



Commutator of two operators

The commutator of two operators \hat{A} and \hat{B} is defined by

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$[\hat{A}, \hat{B}]\phi \equiv \hat{A}\hat{B}\phi - \hat{B}\hat{A}\phi$$

Why is it important?



Commutator of two operators

$\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$

$\hat{B}\hat{A}\phi \equiv \hat{B}(\hat{A}\phi)$

$\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ may not be the same



4

Commutator of two operators

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

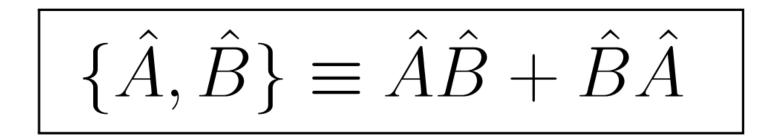
Two operators are said to commute.

$$[\hat{A}, \hat{A}] = 0$$



Anticommutator

The anticommutator $\{\hat{A}, \hat{B}\}$ is defined by





The **position operator** \hat{x} that acting on functions of x gives another function of x as follows:

$$\hat{x}f(x) \equiv xf(x)$$

$$\hat{x}^k f(x) \equiv x^k f(x)$$



Momentum operator

The momentum operator acts on wave functions, which are functions of space and time to give another function of space and time:

$$\hat{p}\Psi(x,t) = p\Psi(x,t)$$

$$\hat{p} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} = -i\hbar \frac{\partial}{\partial x}$$



Momentum operator

Let us consider the wavefunction of a free particle:

$$\Psi(x,t) = e^{i(xp - Et)/\hbar}$$

$$\begin{split} \hat{p}\Psi(x,t) &= -i\hbar \frac{\partial}{\partial x} e^{i(xp-Et)/\hbar} \\ &= (-i\hbar) \left(\frac{ip}{\hbar}\right) e^{i(xp-Et)/\hbar} \\ &= p\Psi(x,t) \end{split}$$



Linear operator

 $\hat{A}(a\phi) \equiv a\hat{A}\phi$ $\hat{A}(\phi_1 + \phi_2) \equiv \hat{A}\phi_1 + \hat{A}\phi_2$ $(\hat{A} + \hat{B})\phi \equiv \hat{A}\phi + \hat{B}\phi$ $\hat{A}\hat{B}\phi \equiv \hat{A}(\hat{B}\phi)$



$$\hat{x} = x \qquad \hat{p} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} = -i\hbar \frac{\partial}{\partial x}$$

We have operators \hat{x} and \hat{p} that are clearly somewhat related. We would like to know their commutator $[\hat{x}, \hat{p}]$. For this we let $[\hat{x}, \hat{p}]$ act on some arbitrary function $\phi(x)$ and then attempt simplification.



$$\begin{aligned} [\hat{x}, \, \hat{p}]\phi(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x})\phi(x) \\ &= \hat{x}\hat{p}\phi(x) - \hat{p}\hat{x}\phi(x) \\ &= \hat{x}(\hat{p}\phi(x)) - \hat{p}(\hat{x}\phi(x)) \\ &= \hat{x}\left(-i\hbar\frac{\partial\phi(x)}{\partial x}\right) - \hat{p}(x\phi(x)) \end{aligned}$$



$$\begin{split} [\hat{x}, \, \hat{p}]\phi(x) &= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \frac{\partial}{\partial x} (x\phi(x)) \\ &= -i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar x \frac{\partial \phi(x)}{\partial x} + i\hbar \phi(x) \\ &= i\hbar \phi(x) \end{split}$$



$$[\hat{x},\,\hat{p}]\phi(x) = i\hbar\,\phi(x)$$

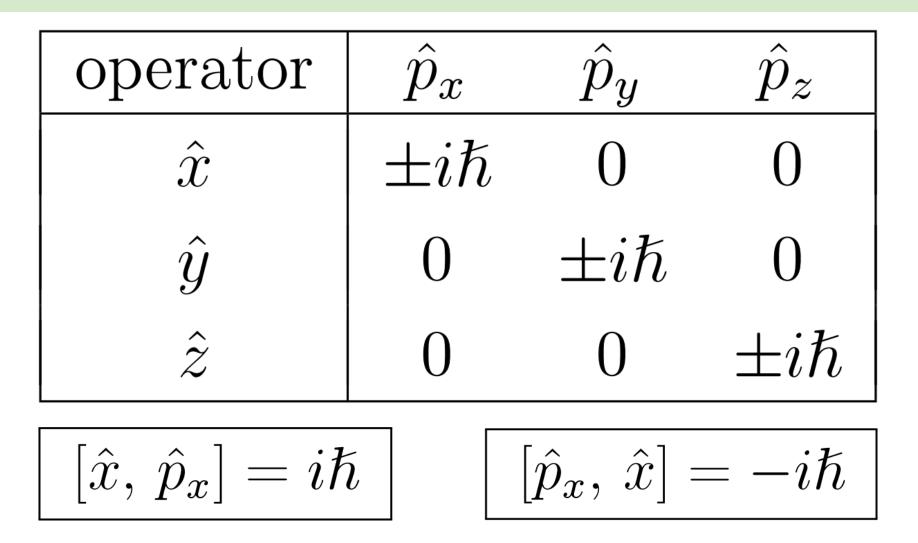
$$[\hat{x}, \hat{p}] = i\hbar$$

$$[\hat{p}, \hat{x}] = -[\hat{x}, \hat{p}] = -i\hbar$$



$\hat{\mathbf{p}} \equiv -i\hbar\nabla$ $\hat{\mathbf{r}} \equiv (\hat{x}, \hat{y}, \hat{z})$ $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$







$$\left[\hat{x}_i, \, \hat{p}_j\right] = i\hbar\delta_{ij}$$

The Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$



$$\hat{p}_k = -i\hbar \frac{\partial}{\partial x_k}, \qquad k = 1, 2, 3.$$

 $(x_1, x_2, x_3) = (x, y, z)$
 $(p_1, p_2, p_3) = (p_x, p_y, p_z)$





operator	position	momentum
1	\hat{x}	\hat{p}_x
2	\hat{y}	\hat{p}_{y}
3	\hat{z}	$\hat{p}_{oldsymbol{z}}$

