Lecture notes: Relativity

Dr. Mohammad A Rashid
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just.edu.bd/t/rashid

1 Inertial frame of reference

A frame of reference in which Newtons first law is valid is called an inertial frame of reference.

2 Postulates of special theory of relativity

In 1905 Einstein introduced the special theory of relativity, proposing drastic revisions in the Newtonian concepts of space and time. The theory is based on two simple postulates.

First Postulate: The laws of physics are the same in every inertial frame of reference.

Second Postulate: The speed of light in vacuum is the same in all inertial frames of reference and is independent of the motion of the source.

Einsteins second postulate immediately implies the following result: It is impossible for an inertial observer to travel at c, the speed of light in vacuum.

3 The Lorentz Coordinate Transformation

Consider an event which is observed in two different frame of references S and S', where the frame S' is moving relative to S with constant speed u in the $+x$ direction. We assume that the origins coincide at the initial time $t = 0 = t'$. The event occurred at point (x, y, z) at time t as observed in frame of reference S. When observed in frame S' it is seen to be occurred at point (x', y', z') at time t' $(\neq t)$.

Assume that x and x' are related as

$$x' = \gamma(x - ut). \tag{1}$$

Here the factor γ does not depend upon either x or t. As the equation of physics must have the same form in both S and S', we write the corresponding equation for x in terms of x' and t':

$$\begin{aligned} x &= \gamma(x' + ut') \\ &= \gamma x' + \gamma u t'. \end{aligned} \tag{2}$$

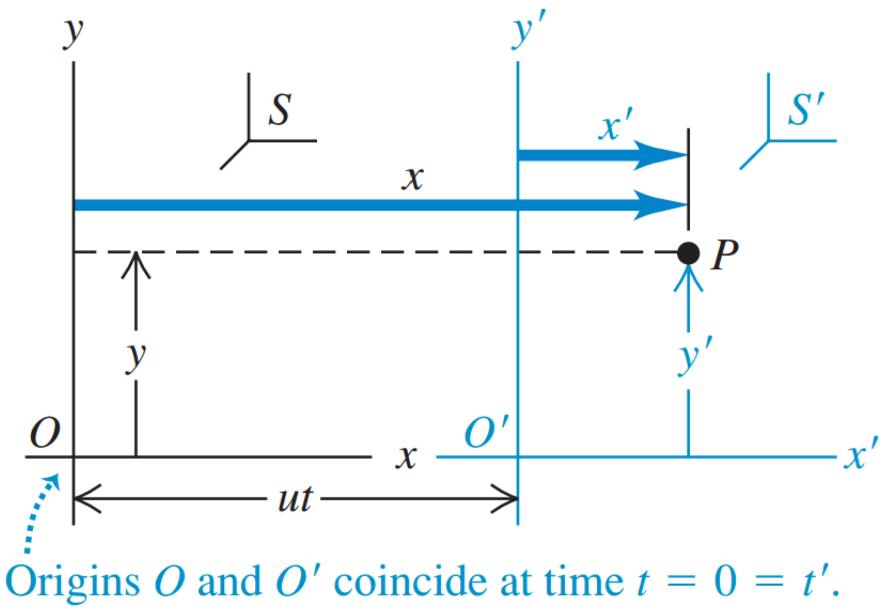

Figure 1: Frame S' moves in the $+x$ direction with the speed u relative to frame S.

The factor γ must be same in both frame of reference since there is no difference between S and S' other than in the sign of u.

As y, y' and z, z' are perpendicular to the direction of u we write

$$y' = y \tag{3}$$
$$z' = z. \tag{4}$$

Inserting Eq. (1) into Eq. (2) we have

$$\begin{aligned}
& x = \gamma^2(x - ut) + \gamma ut' \\
\text{or,} \quad & x = \gamma^2 x - \gamma^2 ut + \gamma ut' \\
\text{or,} \quad & \gamma ut' = x - \gamma^2 x + \gamma^2 ut \\
\text{or,} \quad & \gamma ut' = \gamma^2 ut + (1 - \gamma^2)x \\
\therefore \quad & t' = \gamma t + \left(\frac{1 - \gamma^2}{\gamma u}\right) x
\end{aligned} \tag{5}$$

Equations (1) and (3) to (5) constitute a coordinate transformation that satisfies the first postulate of special theory of relativity.

From the second postulate we know that the speed of light measured in both frames should be same. Suppose a flare is set off at the common origin of S and S' at $t = t' = 0$, and the observers in each frame measure the speed at which the light from flare spreads out. Both observers must find the same speed c, which gives

$$x = ct \tag{6}$$
$$x' = ct' \tag{7}$$

the in frames S and S' respectively.

Substituting the values of x' and t' from Eq. (1) and Eq. (2) respectively into the above equation we have

$$\begin{aligned}
\gamma(x-ut) &= c\gamma t + \left(\frac{1-\gamma^2}{\gamma u}\right) cx \\
\text{or,}\quad \gamma x - u\gamma t &= c\gamma t + \left(\frac{1-\gamma^2}{\gamma u}\right) cx \\
\text{or,}\quad \gamma x - \left(\frac{1-\gamma^2}{\gamma u}\right) cx &= c\gamma t + u\gamma t \\
\text{or,}\quad \gamma x\left[1-\left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{c}{u}\right] &= ct\gamma\left(1+\frac{u}{c}\right) \\
\therefore\quad x &= ct\left[\frac{1+u/c}{1-\left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{c}{u}}\right] \qquad (8)
\end{aligned}$$

Equating both sides of Equations (5) and (6) we have

$$\begin{aligned}
\left[\frac{1+u/c}{1-\left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{c}{u}}\right] &= 1 \\
\text{or,}\quad 1+\frac{u}{c} &= 1-\left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{c}{u} \\
\text{or,}\quad \frac{u}{c} &= -\left(\frac{1-\gamma^2}{\gamma^2}\right)\frac{c}{u} \\
\text{or,}\quad \frac{u^2}{c^2} &= -\frac{1}{\gamma^2}+1 \\
\text{or,}\quad \frac{1}{\gamma^2} &= 1-\frac{u^2}{c^2} \\
\therefore\quad \gamma &= \frac{1}{\sqrt{1-u^2/c^2}} \qquad (9)
\end{aligned}$$

Rewriting Eq. (5)

$$\begin{aligned}
t' &= \gamma\left(t+\frac{1-\gamma^2}{\gamma^2}\frac{x}{u}\right) \\
&= \gamma\left[t+\left(\frac{1}{\gamma^2}-1\right)\frac{x}{u}\right] \\
&= \gamma\left[t+\left(1-\frac{u^2}{c^2}-1\right)\frac{c^2}{u^2}\frac{ux}{c^2}\right] \\
&= \gamma\left(t-\frac{ux}{c^2}\right) \\
&= \frac{t-ux/c^2}{\sqrt{1-u^2/c^2}} \qquad (10)
\end{aligned}$$

Therefore from Equations (1), (3), (4) and (10) we have

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \tag{11}$$

$$y' = y \tag{12}$$

$$z' = z \tag{13}$$

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}} \tag{14}$$

These equations are the Lorentz coordinate transformation, the relativistic generalization of the Galilean coordinate transformation.

3.1 Inverse Lorentz Transformation

$$x = \frac{x' + ut'}{\sqrt{1 - u^2/c^2}} \tag{15}$$

$$y = y' \tag{16}$$

$$z = z' \tag{17}$$

$$t = \frac{t' + ux'/c^2}{\sqrt{1 - u^2/c^2}} \tag{18}$$

4 Mass and Energy, $E = mc^2$

Consider an object which is initially at rest, starts to move due to a force F acting on it. If no other forces act on the object all the work done on it becomes kinetic energy, K:

$$\begin{aligned} K &= W = \text{Work done} \\ &= \int_0^s F\, ds. \end{aligned} \tag{19}$$

In non-relativistic physics $F = d(mv)/dt$ and $K = \frac{1}{2}mv^2$, where m is the mass and v is the speed of the object. To find the correct relativistic formula of K we start from the relativistic form of the second law of motion:

$$\begin{aligned} F &= \frac{d}{dt}(\gamma m v) \\ &= \frac{d}{dt}\left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right) \end{aligned}$$

Hence, from Eq. (19)

$$\begin{aligned} K &= \int_0^s \frac{d(\gamma m v)}{dt}\, ds \\ &= \int_0^{mv} v\, d(\gamma m v) \\ &= \int_0^v v\, d\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \end{aligned}$$

Integrating by parts ($\int x\, dy = xy - \int y\, dx$),

$$\begin{aligned} K &= \frac{mv^2}{\sqrt{1-v^2/c^2}} - m\int_0^v \frac{v\, dv}{\sqrt{1-v^2/c^2}} \\ &= \frac{mv^2}{\sqrt{1-v^2/c^2}} + \left[mc^2\sqrt{1-v^2/c^2}\right]_0^v \\ &= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2 \\ &= \frac{mv^2 + mc^2(1-v^2/c^2)}{\sqrt{1-v^2/c^2}} - mc^2 \\ &= \frac{mv^2 + mc^2 - mv^2}{\sqrt{1-v^2/c^2}} - mc^2 \\ &= \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 \end{aligned}$$

Therefore

$$\text{Kinetic Energy} \qquad K = \gamma mc^2 - mc^2 \tag{20}$$

Equation (20) may be written as

$$\text{Total Energy} \qquad E = \gamma mc^2 = mc^2 + K = E_0 + K \tag{21}$$

where

$$\text{Rest Energy} \qquad E_0 = mc^2 \tag{22}$$

If the object is moving, its total energy is

$$\text{Total Energy} \qquad E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \tag{23}$$

5 Examples

Examples # 01: Verify that

$$\frac{1}{1 - v^2/c^2} = 1 + \frac{p^2}{m^2c^2}$$

Answer

$$\begin{aligned} 1 + \frac{p^2}{m^2c^2} &= 1 + \frac{p^2c^2}{m^2c^4} \\ &= \frac{(mc^2)^2 + (pc)^2}{m^2c^4} \\ &= \frac{E^2}{m^2c^4} \\ &= \frac{(\gamma mc^2)^2}{m^2c^4} \\ &= \gamma^2 \\ &= \frac{1}{1 - v^2/c^2} \end{aligned}$$

References

1. Concepts of Modern Physics by Arthur Beiser
2. Modern Physics `for Scientists and Engineers` by Stephen T. Thornton and Andrew Rex
3. University Physics `with Modern Physics` by Hugh D. Young and Roger A. Freedman