Lecture notes: QM 06

Angular Momentum

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Contents

Angular momentum is as important in classical mechanics as in quantum mechanics. It is particularly useful for studying the dynamics of systems that move under the influence of spherically symmetric, or central potentials. Angular momentum can be of the orbital type, this is the familiar case that occurs when a particle rotates around some fixed point. But is can also be spin angular momentum. This is a different kind of angular momentum and can be carried by point particles. Much of the mathematics of angular momentum is valid both for orbital and spin angular momentum.

Let us begin our analysis of angular momentum by recalling that in three dimensions the usual $\hat{x}$ and $\hat{p}$ operators are vector operators:

$$\begin{aligned} \hat{\mathbf{p}} &= (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\overrightarrow{\nabla} = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \\ \hat{\mathbf{x}} &= (\hat{x}, \hat{y}, \hat{z}). \end{aligned} \tag{1}$$

The commutation relations are as follows:

$$
\begin{aligned}
[\hat{x},\, \hat{p}_x] &= i\hbar \\
[\hat{y},\, \hat{p}_y] &= i\hbar \\
[\hat{z},\, \hat{p}_z] &= i\hbar
\end{aligned} \tag{2}
$$

All other commutators are involving the three coordinates and the three momenta are zero!

1 Angular momentum operator

In classical physics the angular momentum of a particle with momentum $\mathbf{p}$ and position $\mathbf{r}$ is defined by

$$
\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{3}
$$

Hence the components of $\mathbf{L} = (L_x, L_y, L_z)$ are given by

$$
\begin{aligned}
L_x &= y p_z - z p_y, \\
L_y &= z p_x - x p_z, \\
L_z &= x p_y - y p_x.
\end{aligned} \tag{4}
$$

The angular momentum operator $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ can be obtained at once by replacing $\mathbf{r}$ and $\mathbf{p}$ by the corresponding operators in the position representation:

$$
\begin{aligned}
\hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \\
\hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \\
\hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x.
\end{aligned} \tag{5}
$$

In crafting this definition we saw no ordering ambiguities. Each angular momentum operator is the difference of two terms, each term consisting of a product of a coordinate and a momentum. But note that in all cases it is a coordinate and a momentum along different axes, so they commute. Had we written $\hat{L}_x = \hat{p}_z\hat{y} - \hat{p}_y\hat{z}$, it would have not mattered, it is the same as the $\hat{L}_x$ above. It is simple to check that the angular momentum are Hermitian.Take $\hat{L}_x$ for example. Recalling that for any two operators $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$ we have

$$
(\hat{L}_x)^\dagger = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)^\dagger = (\hat{y}\hat{p}_z)^\dagger - (\hat{z}\hat{p}_y)^\dagger = (\hat{p}_z)^\dagger(\hat{y})^\dagger - (\hat{p}_y)^\dagger(\hat{z})^\dagger. \tag{6}
$$

Since all coordinates and momenta are Hermitian operators, we have

$$
(\hat{L}_x)^\dagger = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x, \tag{7}
$$

where we moved the momenta to the right of the coordinates by virtue of vanishing commutators. The other two angular momentum operators are also Hermitian, so we have

$$
\boxed{\hat{L}_x^\dagger = \hat{L}_x, \qquad \hat{L}_y^\dagger = \hat{L}_y, \qquad \hat{L}_z^\dagger = \hat{L}_z.} \tag{8}
$$

Clearly, angular momentum does not exist in a one-dimensional space. We should mention that the components $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, and the square of $\hat{\mathbf{L}}$,

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \tag{9}$$

all are Hermitian. Hence all the angular momentum operators are observables and have real eigenvalues.

2 Commutation relations

Given a set of Hermitian operators, it is natural to ask what are their commutators. This computation enables us to see if we can measure them simultaneously. Let us compute the commutator of $\hat{L}_x$ with $\hat{L}_y$. Since $\hat{x}$, $\hat{y}$, and $\hat{z}$ mutually commute and so $\hat{p}_x$, $\hat{p}_y$, and $\hat{p}_z$ we have using (2)

$$\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}\hat{p}_y, \hat{p}_z] \\
&= \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y \\
&= \hat{y}(-i\hbar)\hat{p}_x + \hat{x}(i\hbar)\hat{p}_y \\
&= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \\
&= i\hbar\hat{L}_z.
\end{aligned} \tag{10}$$

A similar calculation yields the other two commutation relations; but it is much simpler to infer them from (10) by means of a cyclic permutation of the xyz components, $x \to y \to z \to x$:

$$\boxed{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y.} \tag{11}$$

This is the full set of commutators of angular momentum operators. The set is referred to as the **algebra of angular momentum**. Notice that while the operators $\hat{L}$ were defined in terms of coordinates and momenta, the final answer for the commutators do not involve coordinates nor momenta: commutators of angular momenta give angular momenta! The $\hat{L}$ operators are also referred to as *orbital* angular momentum, to distinguish them from spin angular momentum operators. The spin angular momentum operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ cannot be written in terms of coordinates and momenta. They are more abstract entities, in fact their simplest representation is as two-by-two matrices! Still, being angular momenta they satisfy exactly the same algebra as their orbital cousins. We have

$$\boxed{[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \qquad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \qquad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y.} \tag{12}$$

We have seen that the commutator $[\hat{x}, \hat{p}] = i\hbar$ is associated with the fact that we cannot have simultaneous eigenstates of position and of momentum. Let us now see what the commutators of angular momentum operators tell us. In particular: can we have simultaneous eigenstates of $\hat{L}_x$ and $\hat{L}_y$? As it turns out, the answer is no, we cannot. To demonstrate this let us assume that there exists a wave function ψ_0 which is simultaneously an eigenstate of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ such that

$$\begin{aligned} \hat{L}_x\psi_0 &= \lambda_x\psi_0, \\ \hat{L}_y\psi_0 &= \lambda_y\psi_0, \\ \hat{L}_z\psi_0 &= \lambda_z\psi_0. \end{aligned} \tag{13}$$

Letting the first commutator identity of (11) act on ψ_0 we have

$$\begin{aligned} i\hbar\hat{L}_z\psi_0 &= [\hat{L}_x,\, \hat{L}_y]\psi_0 \\ &= \hat{L}_x\hat{L}_y\psi_0 - \hat{L}_y\hat{L}_x\psi_0 \\ &= \hat{L}_x\lambda_y\psi_0 - \hat{L}_y\lambda_x\psi_0 \\ &= (\lambda_x\lambda_y - \lambda_y\lambda_x)\psi_0 \\ &= 0, \end{aligned} \tag{14}$$

showing that $\lambda_z = 0$. But this is not all, looking at the other commutators in the angular momentum algebra we see that they also vanish acting on ψ_0 and as a result λ_x and λ_y must be zero.

All in all, assuming that ψ_0 is a simultaneous eigenstate of $\hat{L}_x$, $\hat{L}_y$, and $\hat{L}_z$ has led to $\hat{L}_x\psi_0 = \hat{L}_y\psi_0 = \hat{L}_z\psi_0 = 0$. The state is annihilated by *all* angular momentum operators. This trivial situation is not very interesting. We have learned that it is impossible to find states that are nontrivial simultaneous eigenstates of any two of the angular momentum operators.

For commuting Hermitian operators, there is no problem finding simultaneous eigenstates. In fact, commuting Hermitian operators always have a complete set of simultaneous eigenstates. Suppose we select $\hat{L}_z$ as one of the operators we want to measure. Can we now find a second Hermitian operator that commutes with it? The answer is yes. As it turns out, $\hat{\mathbf{L}}^2$, defined in (9) commutes with $\hat{L}_z$ and is an interesting choice for a second operator. Indeed

$$\begin{aligned} [\hat{L}_z,\, \hat{\mathbf{L}}^2] &= [\hat{L}_z,\, \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z] \\ &= [\hat{L}_z,\, \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y] \\ &= [\hat{L}_z,\, \hat{L}_x]\hat{L}_x + \hat{L}_x[\hat{L}_z,\, \hat{L}_x] + [\hat{L}_z,\, \hat{L}_y]\hat{L}_y + \hat{L}_y[\hat{L}_z,\, \hat{L}_y] \\ &= i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_x \\ &= 0. \end{aligned} \tag{15}$$

So we should be able to find simultaneous eigenstates of both $\hat{L}_z$ and $\hat{\mathbf{L}}^2$. The operator $\hat{\mathbf{L}}^2$ is **Casimir operator**, which means that it commutes with all angular momentum operators. Just like it commutes with $\hat{L}_z$, it commutes also with $\hat{L}_x$ and $\hat{L}_y$.

3 Angular momentum in spherical coordinates

To understand the angular momentum operators a little better, let us write them in spherical coordinates. For this we need the relation between (r, θ, ϕ) and the Cartesian coordinates (x, y, z):

$$\begin{aligned}
x &= r\sin\theta\cos\phi, & r &= \sqrt{x^2+y^2+z^2}, \\
y &= r\sin\theta\sin\phi, & \theta &= \cos^{-1}\left(\frac{z}{r}\right), \\
z &= r\cos\theta, & \phi &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned} \tag{16}$$

The angular momentum operators generate rotations. In spherical coordinates rotations about the z axis are the simplest: they change ϕ but leave θ invariant. Both rotations about the x and y axes change θ and ϕ. We can therefore hope that $\hat{L}_z$ is simple in spherical coordinates. Using the definition $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ we have

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{17}$$

Notice that this is related to $\frac{\partial}{\partial\phi}$ since, by the chain rule

$$\begin{aligned}
\frac{\partial}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z} \\
&= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0 \\
&= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},
\end{aligned} \tag{18}$$

where we have used (16) to evaluate the partial derivatives. Using the last two equations we can identify

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}.} \tag{19}$$

This is a very simple and useful representation. It confirms the interpretation that $\hat{L}_z$ generates rotations about the z axis, as it has to do with changes of ϕ. Note that $\hat{L}_z$ is like a momentum along the "circle" defined by the ϕ coordinate ($\phi = \phi + 2\pi$). The other angular momentum operators are

$$\begin{aligned}
\hat{L}_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \\
\hat{L}_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right).
\end{aligned} \tag{20}$$

With a longer calculation it can be show that

$$\boxed{\hat{\mathbf{L}}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].} \tag{21}$$

4 Eigenvalues of angular momentum

We demonstrated before that the Hermitian operators $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ commute. We now aim to constructthe simultaneous eigenfunctions of these operators. They will be functions of θ and ϕ and we will call them $\psi_{lm}(\theta,\phi)$. The conditions that they be eigenfunctions are

$$\begin{aligned} \hat{L}_z\,\psi_{lm}(\theta,\phi) &= \hbar m\,\psi_{lm}(\theta,\phi), & m \in \mathbb{R} \\ \hat{\mathbf{L}}^2\,\psi_{lm}(\theta,\phi) &= \hbar^2\, l(l+1)\,\psi_{lm}(\theta,\phi), & l \in \mathbb{R}. \end{aligned} \tag{22}$$

As befits Hermitian operators, the eigenvalues are real. Both m and l are unit free; there is an $\hbar$ in the $\hat{L}_z$ eigenvalue because angular momentum has units of $\hbar$. For the eigenvalue of $\hat{\mathbf{L}}^2$ we have an $\hbar^2$. Note that we have written the eigenvalue of $\hbar^2$ as $l(l+1)$ and for l real this is always greater than orequal to $-1/4$. In fact $l(l+1)$ ranges from zero to infinity as l ranges from zero to infinity. We can show that the eigenvalues of $\hat{\mathbf{L}}^2$ can't be negative. For this we first claim that

$$(\psi,\, \hat{\mathbf{L}}^2\psi) \geq 0, \tag{23}$$

and taking ψ to be a normalized eigenfunction with $\hat{\mathbf{L}}^2$ eigenvalue λ we immediately see that the above gives $(\psi,\, \lambda\psi) = \lambda \geq 0$, as desired. To prove the above equation we simply expand and use Hermiticiy:

$$\begin{aligned} (\psi,\, \hat{\mathbf{L}}^2\psi) &= (\psi,\, \hat{L}_x^2\psi) + (\psi,\, \hat{L}_y^2\psi) + (\psi,\, \hat{L}_z^2\psi) \\ &= (\hat{L}_x\psi,\, \hat{L}_x\psi) + (\hat{L}_y\psi,\, \hat{L}_y\psi) + (\hat{L}_z\psi,\, \hat{L}_z\psi) \geq 0, \end{aligned} \tag{24}$$

because each of the three summands is greater than or equal to zero.

Let us now solve the first eigenvalue equation in (22) using the coordinate representation (16) for $\hat{L}_z$ operator:

$$-i\hbar\frac{\partial\psi_{lm}}{\partial\phi} = \hbar m\,\psi_{lm} \qquad \rightarrow \qquad \frac{\partial\psi_{lm}}{\partial\phi} = im\,\psi_{lm}. \tag{25}$$

This determines the ϕ dependence of the solution and we write

$$\psi_{lm}(\theta,\phi) = e^{im\phi}\,P_l^m(\theta), \tag{26}$$

where the function $P_l^m(\theta)$ captures the still undetermined θ dependence of the eigenfunction ψ_{lm}. We will require that ψ_{lm} be uniquely defined as a function of the angles and this requires that[1]

$$\psi_{lm}(\theta,\phi+2\pi) = \psi_{lm}(\theta,\phi). \tag{27}$$

There is no similar condition for θ. The above condition requires that

$$e^{im(\phi+2\pi)} = e^{im\phi} \qquad \rightarrow \qquad e^{i2\pi m} = 1. \tag{28}$$

[1]One may have tried to require that after ψ increases by 2π the wavefunction changes sign, but this does not lead to a consistent set of ψ_{lm}'s.

This equation implies that m must be an integer:

$$\boxed{m \in \mathbb{Z}} \tag{29}$$

This completes our analysis of the first eigenvalue equation. With a longer calculation it can be shown that the possible values of l are

$$l = 0,\ 1,\ 2,\ 3,\ \ldots \qquad \text{and} \qquad -l \leq m \leq l. \tag{30}$$

One can think of the ψ_{lm} eigenfunctions as first determined by the integer l and, for a fixed l, there are $2l + 1$ choices of m: $-l$, $-l + 1$, ..., l. Our ψ_{lm} eigenfunctions, with suitable normalization, are called the **spherical harmonics** $Y_{lm}(\theta, \phi)$.

5 Eigenfunctions of angular momentum

We demonstrated before that the Hermitian operators $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ commute. Therefore there will be a simultaneous eigenfunctions of these operators. They will be functions of θ and ϕ and we write them as $Y_{lm}(\theta, \phi)$, are called the **spherical harmonics**. The properly normalized spherical harmonics are

$$\boxed{Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\left(\frac{2l+1}{4\pi}\right) \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}, \qquad (m \geq 0).} \tag{31}$$

Here $P_l^m(\cos\theta)$ is the associated Legendre functions. $m < 0$, we use

$$Y_{l,m}(\theta, \phi) = (-1)^m [Y_{l,-m}(\theta, \phi)]^*. \tag{32}$$

The eigenvalues of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ corresponding to the eigenfunction $Y_{l,m}(\theta, \phi)$ can be known from the following eigenvalue equations:

$$\begin{aligned} \hat{L}_z Y_{lm} &= \hbar m Y_{lm}, \qquad (m \in \mathbb{Z}) \\ \hat{\mathbf{L}}^2 Y_{lm} &= \hbar^2 l(l+1) Y_{lm}, \end{aligned} \tag{33}$$

with

$$l = 0, 1, 2, 3, \ldots \qquad \text{and} \qquad -l \leq m \leq l. \tag{34}$$

The first few spherical harmonics are given below.

Being eigenstates of Hermitian operators with different eigenvalues, spherical harmonics with different l and m subscripts are automatically orthogonal. The complicated normalization factor is needed to make them have unit normalization. The spherical harmonics form an orthonormal set with respect to integration over the solid angle.

$$\boxed{\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \, \sin\theta \, Y^*_{l'm'}(\theta, \phi)\, Y_{lm}(\theta, \phi) = (Y^*_{l'm'},\, Y_{lm}) = \delta_{l'l}\, \delta_{m'm}.} \tag{35}$$

$Y_{lm}(\theta, \varphi)$	$Y_{lm}(x, y, z)$
$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{10}(x, y, z) = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
$Y_{1,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$	$Y_{1,\pm1}(x, y, z) = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$
$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$	$Y_{20}(x, y, z) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$
$Y_{2,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta\cos\theta$	$Y_{2,\pm1}(x, y, z) = \mp\sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$	$Y_{2,\pm2}(x, y, z) = \sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$

6 Central potential

Consider a particle represented by a three-dimensional wave function $\psi(x, y, z)$ moving in a three dimensional potential $V(\mathbf{r})$. The Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{36}$$

We have a **central potential** if $V(\mathbf{r}) = V(r)$. A central potential has no angular dependence, the value of the potential depends only on the distance r from the origin. A central potential is spherically symmetric; the surfaces of constant potential are spheres centered at the origin and it is therefore rotationally invariant. The equation above for a central potential is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{37}$$

Note that the wavefunction is a full function of $\mathbf{r}$, it will only be rotational invariant for the simplest kinds of solutions. Given the rotational symmetry of the potential we are led to express the Schrödinger equation and energy eigenfunctions using spherical coordinates.

In spherical coordinates, the Laplacian is

$$\nabla^2\psi = (\nabla\cdot\nabla)\psi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi. \tag{38}$$

Therefor the Schrödinger equation for a particle in a central potential becomes

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right]\psi + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

The angular dependent piece of the ∇^2 operator is the magnitude squared of the angular momentum operator. Hence the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\hat{\mathbf{L}}^2}{\hbar^2}\right]\psi(\mathbf{r}) + V(r)\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{39}$$

or

$$
\boxed{-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi(\mathbf{r})+\frac{\hat{\mathbf{L}}^2}{2mr^2}\psi(\mathbf{r})+V(r)\psi(\mathbf{r})=E\psi(\mathbf{r}).} \tag{40}
$$

The wave function will have two parts: one will be the radial part and is dependent only on r and the other part is the angle dependent part. The angle dependent is known as the spherical harmonics, $Y_{lm}(\theta,\phi)$.

Note: Most of the materials in this lecture note are taken from the lecture on Quantum Physics by Prof. Barton Zwiebach for the course 8.04 in the year of 2016 at MIT, USA.

References

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