Normalization of Wave Function

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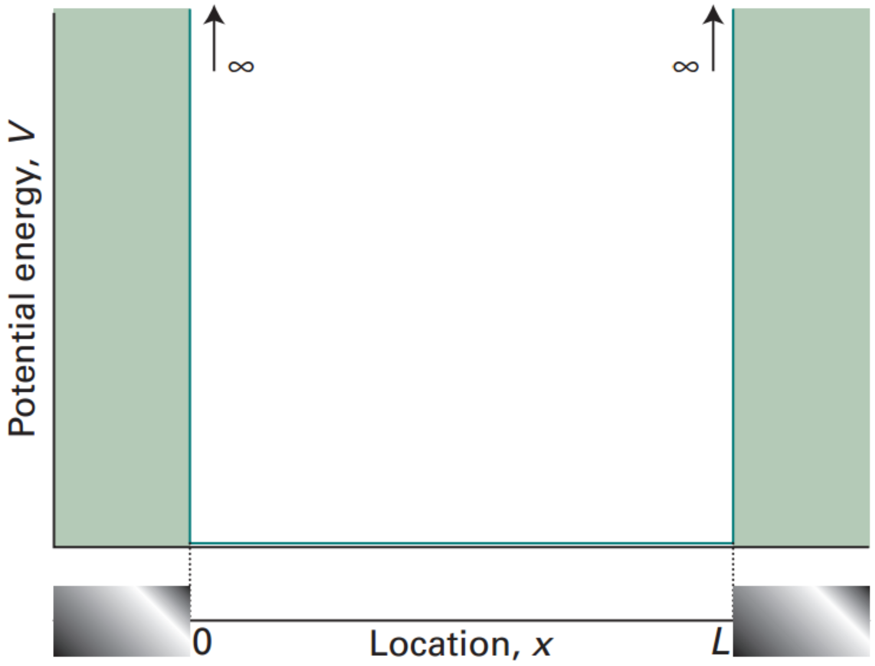
Infinite Square-Well Potential

$$V(x) = \begin{cases} \infty & x \leq 0,\ x \geq L \\ 0 & 0 < x < L \end{cases}$$



Infinite Square-Well Potential

One dimensional time-independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)+V(x)\psi(x)=E\psi(x)$$

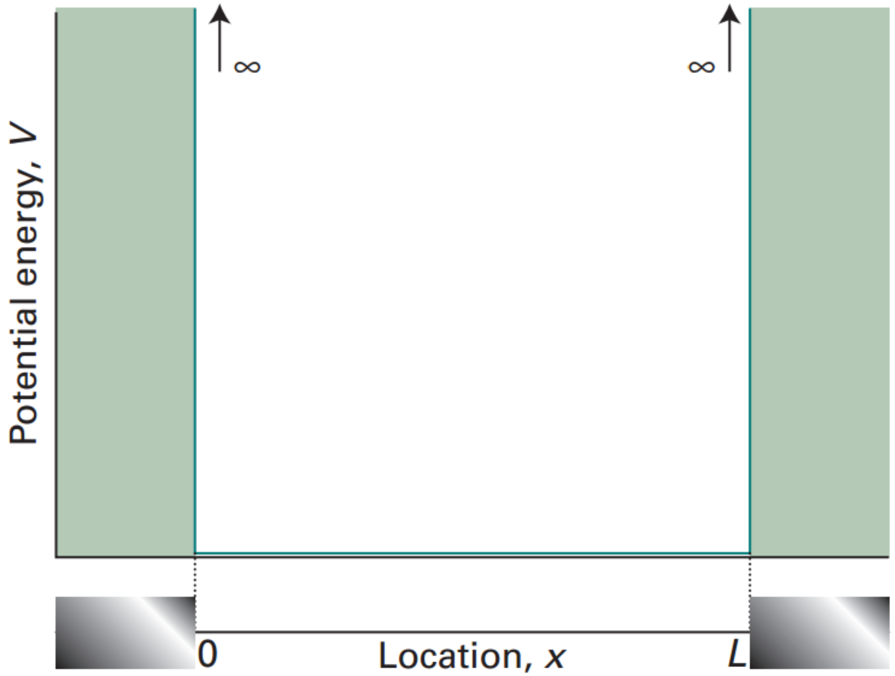
$$\boxed{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}=E\psi(x)}$$

Infinite Square-Well Potential

$$\psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}$$

$$(n = 1, 2, 3, 4, \ldots)$$



Infinite Square-Well Potential

To find the value of A we normalize the wave function over the total distance $-\infty < x < \infty$.

$$\int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)\mathrm{d}x = 1$$

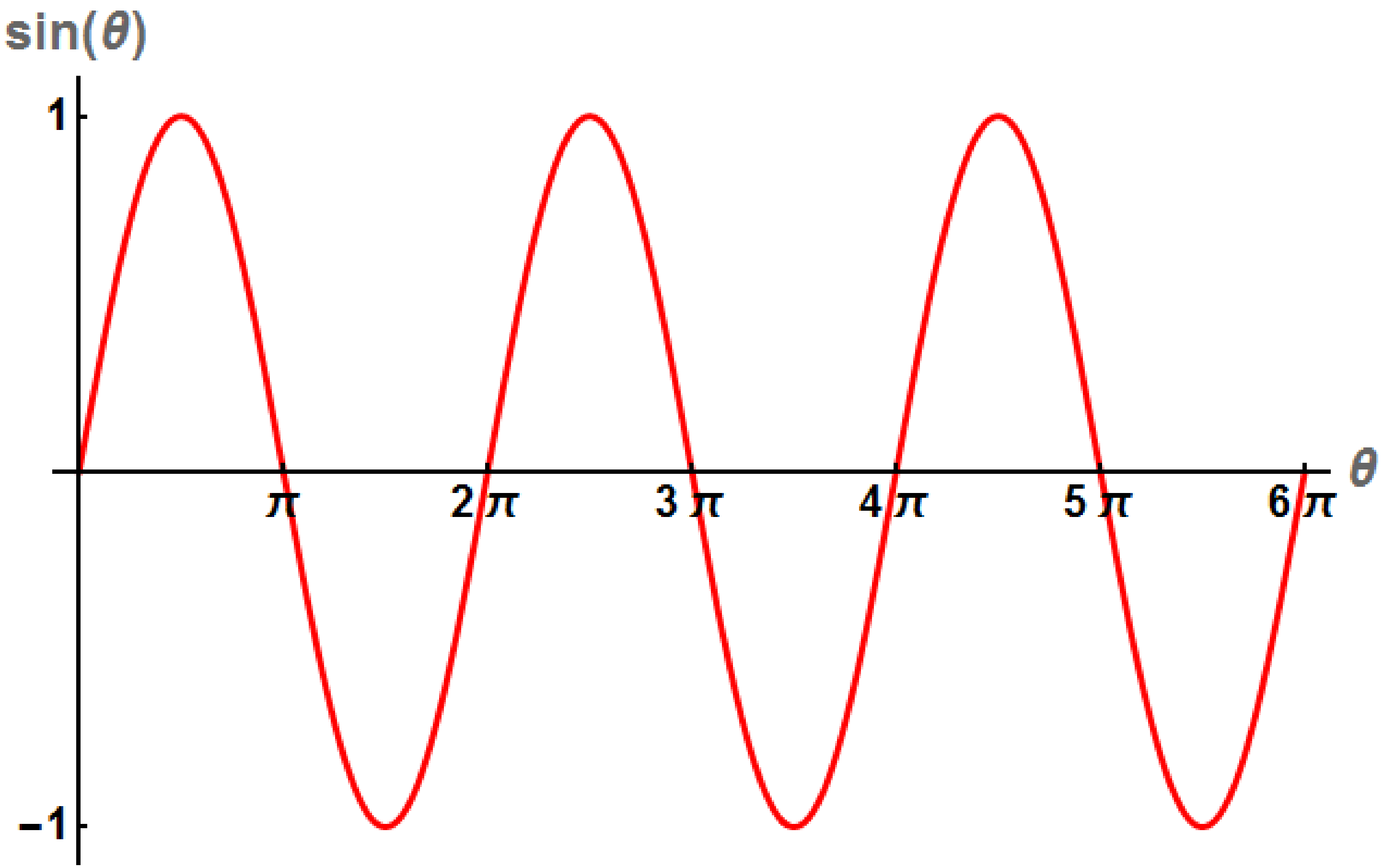
Substitution of the wave function yields

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 1$$

$$
\begin{aligned}
\int \sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= \frac{1}{2}\int 2\sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \frac{1}{2}\int \left[1 - \cos\left(2\frac{n\pi x}{L}\right)\right] \mathrm{d}x \\
&= \frac{1}{2}\left[\int \mathrm{d}x - \int \cos\left(\frac{2n\pi x}{L}\right) \mathrm{d}x\right] \\
&= \frac{1}{2}\left[x - \frac{L}{2n\pi}\int \cos\left(\frac{2n\pi x}{L}\right) \mathrm{d}\left(\frac{2n\pi x}{L}\right)\right] \\
&= \frac{1}{2}\left[x - \frac{L}{2n\pi} \times \sin\left(\frac{2n\pi x}{L}\right)\right]
\end{aligned}
$$

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x \;=\; \frac{1}{2}\left[x - \frac{L}{2n\pi} \times \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L$$

$$(n = 1, 2, 3, 4, \ldots)$$



$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{1}{2}\left[x - \frac{L}{2n\pi} \times \sin\left(\frac{2n\pi x}{L}\right)\right]_0^L$$

$$= \frac{L}{2}$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x = A^2 \frac{L}{2}$$

$$\boxed{A = \sqrt{\frac{2}{L}}}$$

Normalized Wave Function

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)}$$

$$(n = 1, 2, 3, 4, \ldots)$$

Infinite Square-Well Potential

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

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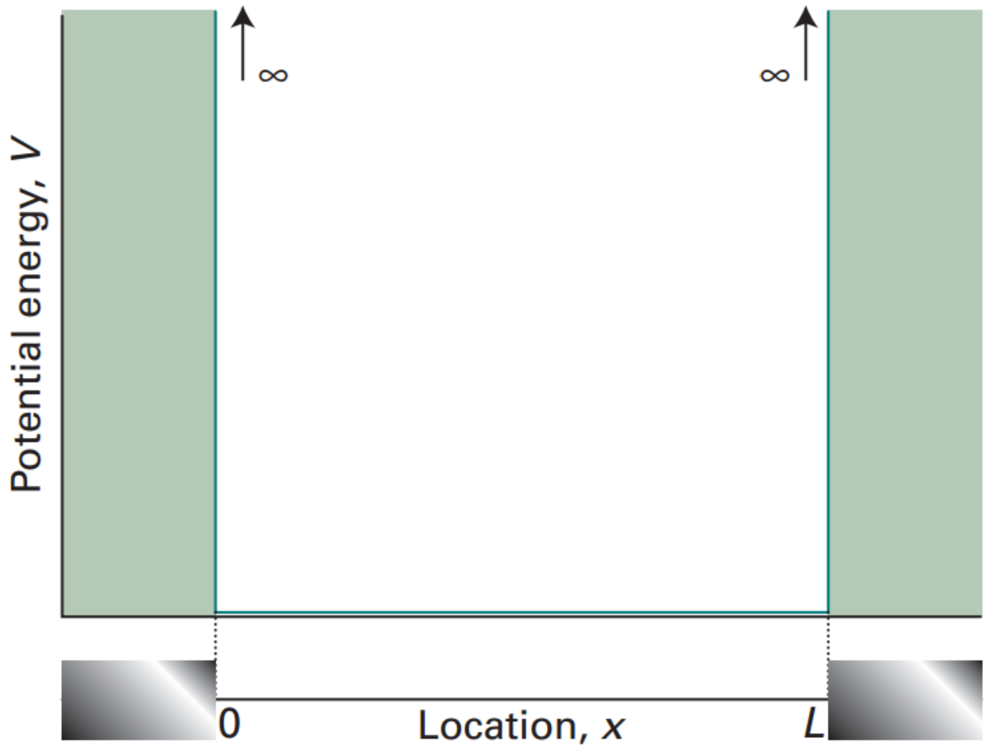
$$(n = 1, 2, 3, 4, \ldots)$$



The Expectation Value

Postulate 4. For a system in a state described by a normalized wave function Ψ , the average or expectation value of the observable corresponding to A is given by

$$\langle A \rangle = \int \Psi^* \, \hat{A} \, \Psi \, d\tau$$

QM05: Postulates of Quantum Mechanics [বাংলা]

https://youtu.be/DZ8LFkJkGKo

The Expectation Value of Position

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2 \, dx = \frac{2}{L} \int_0^L x \sin^2 \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{L} \left[\frac{x^2}{4} - \frac{x \sin(2n\pi x/L)}{4n\pi/L} - \frac{\cos(2n\pi x/L)}{8(n\pi/L)^2} \right]_0^L$$

$$= \frac{2}{L}\left(\frac{L^2}{4}\right) = \frac{L}{2}$$

The Expectation Value of Momentum

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \hat{p} \psi \, dx = \int_{-\infty}^{\infty} \psi^* \left(\frac{\hbar}{i} \frac{d}{dx} \right) \psi \, dx$$

$$= \frac{\hbar}{i} \frac{2}{L} \frac{n\pi}{L} \int_0^L \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{\hbar}{iL} \left[\sin^2 \frac{n\pi x}{L} \right]_0^L = 0$$

The Expectation Value of Momentum

$$E = p^2/2m$$

$$p_n = \pm\sqrt{2mE_n} = \pm\frac{n\pi\hbar}{L}$$

$$p_{\text{av}} = \frac{(+n\pi\hbar/L) + (-n\pi\hbar/L)}{2} = 0$$

The Expectation Value of Momentum

$$\hat{p}\psi_n = p_n\psi_n$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}e^{i\theta} - \frac{1}{2i}e^{-i\theta}$$

The Expectation Value of Momentum

Momentum eigenfunctions for trapped particle

$$\psi_n^{+} = \frac{1}{2i}\sqrt{\frac{2}{L}}\,e^{in\pi x/L}$$

$$\psi_n^{-} = \frac{1}{2i}\sqrt{\frac{2}{L}}\,e^{-in\pi x/L}$$

The Expectation Value of Momentum

$$\hat{p}\psi_n^+ = p_n^+\psi_n^+$$

$$p_n^+ = +\frac{n\pi\hbar}{L}$$

Similarly the wave function ψ_n^- leads to

$$p_n^- = -\frac{n\pi\hbar}{L}$$

Determine the expectation values for x, x^2, p, and p^2 of a particle in an infinite square well for the first excited state.

The first excited state corresponds to $n = 2$,

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

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