Examples of Hermitian operator

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Hermitian operator

An operator $\hat{\Omega}$, which corresponds to a physical observable Ω, is said to be Hermitian if

$$\int \Phi^* \hat{\Omega} \Psi \, \mathrm{d}x = \int (\hat{\Omega} \Phi)^* \Psi \, \mathrm{d}x$$

Properties of Hermitian operator

- The expectation value of a Hermitian operator is real.
- The eigenvalues of a Hermitian operator are real.
- The eigenfunctions can be organized to satisfy orthonormality.
- The eigenfunctions of Hermitian operator form a complete set of basis functions. Any reasonable wave function can be written as a superposition of eigenfunctions of that operator.

Operators

Every observable in quantum mechanics is represented by a linear, Hermitian operator.

$$\hat{x} = x$$

$$\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{\Omega}\Psi(x) = \omega\Psi(x)$$

Briefer notation

The integrals of pairs of functions:

$$(\Phi, \Psi) = \int \Phi^*(x)\Psi(x)\,\mathrm{d}x$$

For any complex constant *a*:

$$(a\Phi, \Psi) = a^*(\Phi, \Psi)$$

$$(\Phi, a\Psi) = a(\Phi, \Psi)$$

Hermitian operator

Hermitian: $$(\Phi, \hat{\Omega}\Psi) = (\hat{\Omega}\Phi, \Psi)$$

Anti-Hermitian: $$(\Phi, \hat{\Omega}\Psi) = -(\hat{\Omega}\Phi, \Psi)$$

Position operator

01: Position operator $\hat{x}$ is a Hermitian operator.

$$\begin{aligned}(\Phi,\ \hat{x}\Psi) &= \int \Phi^*(x)\big(\hat{x}\Psi(x)\big)\,\mathrm{d}x \\ &= \int \Phi^*(x)\big(x\Psi(x)\big)\,\mathrm{d}x \\ &= \int \big(x\Phi^*(x)\big)\Psi(x)\,\mathrm{d}x\end{aligned}$$

$$\hat{x}\Phi(x) = x\Phi(x)$$
$$\hat{x}\Psi(x) = x\Psi(x)$$

Position operator

$$
\begin{aligned}
(\Phi,\, \hat{x}\Psi) &= \int \left(x\Phi(x)\right)^* \Psi(x)\, \mathrm{d}x \\
&= \int \left(\hat{x}\Phi(x)\right)^* \Psi(x)\, \mathrm{d}x \\
&= (\hat{x}\Phi,\, \Psi)
\end{aligned}
$$

Position operator $\hat{x}$ is a Hermitian operator.

Momentum operator

02: Momentum operator $\hat{p}$ is a Hermitian operator.

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\begin{aligned}(\Phi, \hat{p}\Psi) &= \int \Phi^*(x)\left(\hat{p}\Psi(x)\right)\mathrm{d}x \\ &= \int \Phi^*(x)\,(-i\hbar)\frac{\partial\Psi(x)}{\partial x}\,\mathrm{d}x \\ &= -i\hbar\int \Phi^*(x)\,\frac{\partial\Psi(x)}{\partial x}\,\mathrm{d}x\end{aligned}$$

Momentum operator

Integrating by parts we have

$$(\Phi,\, \hat{p}\Psi) = -i\hbar \int \Phi^*(x)\, \frac{\partial \Psi(x)}{\partial x}\, \mathrm{d}x$$

$$= -i\hbar \Big[\Phi^*(x)\Psi(x) \Big]_{-\infty}^{\infty} - (-i\hbar) \int \frac{\partial \Phi^*(x)}{\partial x}\, \Psi(x)\, \mathrm{d}x$$

Since the wave function vanishes as $x \to \pm\infty$ the first term in the right-hand side is zero.

Momentum operator

$$
\begin{aligned}
(\Phi,\, \hat{p}\Psi) &= i\hbar \int \frac{\partial \Phi^*(x)}{\partial x}\, \Psi(x)\, \mathrm{d}x \\
&= \int \left(-i\hbar \frac{\partial \Phi(x)}{\partial x} \right)^* \Psi(x)\, \mathrm{d}x \\
&= \int \left(\hat{p}\Phi(x) \right)^* \Psi(x)\, \mathrm{d}x \\
&= (\hat{p}\Phi,\, \Psi)
\end{aligned}
$$

Momentum operator $\hat{p}$ is a Hermitian operator.

Hermitian operator

$$\frac{\partial}{\partial x} \text{ is an anti-Hermitian operator.}$$

$$\frac{\partial^2}{\partial x^2} \text{ is a Hermitian operator.}$$

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