Time dependence of expectation values

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Schrödinger Equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t)$$

For a particle quantum mechanical of mass m moving in a potential $V(x,t)$ the Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$$

Hermitian operator

An operator $\hat{\Omega}$, which corresponds to a physical observable Ω, is said to be Hermitian if

$$\boxed{\int \Phi^* \hat{\Omega} \Psi \, \mathrm{d}x = \int (\hat{\Omega} \Phi)^* \Psi \, \mathrm{d}x}$$

Examples: position operator $\hat{x}$, momentum operator $\hat{p}$, Hamiltonian operator $\hat{H}$, etc.

Expectation value

The expectation value $\langle \Omega \rangle$ of any operator $\hat{\Omega}$:

$$\langle \Omega \rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\,\hat{\Omega}\Psi(x,t)\,\mathrm{d}x$$

The physical meaning of this would be if we consider many copies of the identical system, and measure Ω at a time t in all of them, then the average value recorded will converge to $\langle \Omega \rangle$ as the number of systems and measurements approaches infinity.

Time dependence of expectation values

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\langle \Omega \rangle = i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \Psi^*(x,t)\, \hat{\Omega} \Psi(x,t)\, \mathrm{d}x$$

$$= i\hbar \int_{-\infty}^{\infty} \left(\frac{\partial \Psi^*}{\partial t} \hat{\Omega} \Psi + \Psi^* \hat{\Omega} \frac{\partial \Psi}{\partial t} \right) \mathrm{d}x$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = \left(\hat{H}\Psi\right)^*$$

Time dependence of expectation values

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\Omega\rangle = \int_{-\infty}^{\infty}\left\{-(\hat{H}\Psi)^*\,\hat{\Omega}\Psi + \Psi^*\,\hat{\Omega}(\hat{H}\Psi)\right\}\mathrm{d}x$$

$$= \int_{-\infty}^{\infty}\left\{\Psi^*\,\hat{\Omega}(\hat{H}\Psi) - (\hat{H}\Psi)^*\,\hat{\Omega}\Psi\right\}\mathrm{d}x$$

Hermitian operator: $$\int \Phi^*\hat{\Omega}\Psi\,\mathrm{d}x = \int (\hat{\Omega}\Phi)^*\Psi\,\mathrm{d}x$$

Time dependence of expectation values

$$
\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\Omega\rangle &= \int_{-\infty}^{\infty}\left(\Psi^*\,\hat{\Omega}\hat{H}\Psi - \Psi^*\,\hat{H}\hat{\Omega}\Psi\right)\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\Psi^*(\hat{\Omega}\hat{H} - \hat{H}\hat{\Omega})\Psi\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\Psi^*[\hat{\Omega},\ \hat{H}]\Psi\,\mathrm{d}x
\end{aligned}
$$

Commutator of two operators: $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$

Time dependence of expectation values

We have proven that for operators $\hat{\Omega}$ that do not explicitly depend on time,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\langle \Omega \rangle = \left\langle [\hat{\Omega},\ \hat{H}] \right\rangle$$

Therefore, for an operation which commute with the Hamiltonian operator $\hat{H}$ the expectation value will not change over time.

Time dependence of expectation values

Example: Consider the momentum operator $\hat{p}$ for a free particle.

For a free particle the Hamiltonian operator: $\hat{H} = \dfrac{\hat{p}^2}{2m}$

Time dependence of expectation values

Hence the momentum operator $\hat{p}$ commute with $\hat{H}$:

$$\begin{aligned}[\hat{p},\ \hat{H}] &= [\hat{p},\ \frac{\hat{p}^2}{2m}] = \frac{1}{2m}[\hat{p},\ \hat{p}^2] \\ &= \frac{1}{2m}\Big([\hat{p},\ \hat{p}]\hat{p} + \hat{p}[\hat{p},\ \hat{p}]\Big) = 0\end{aligned}$$

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

Time dependence of expectation values

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle p\rangle = \left\langle [\hat{p},\ \hat{H}] \right\rangle = 0$$

For a free particle the expectation value of momentum does not change over time.

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