Quantum Harmonic Oscillator

Factorizing Hamiltonian | Ground State Wave Function & Energy

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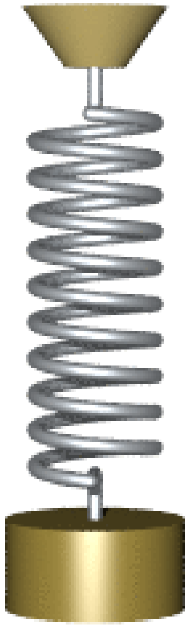
Simple Harmonic Oscillator

It is one of those few problems that are important to all branches of physics. The harmonic oscillator provides a useful model for a variety of vibrational phenomena that are encountered, for instance, in classical mechanics, electrodynamics, statistical mechanics, solid state, atomic, nuclear, and particle physics. In quantum mechanics, it serves as an invaluable tool to illustrate the basic concepts and the formalism.



The Hamiltonian of SHO

The total energy $$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Here $\omega = \sqrt{k/m}$ is angular frequency of the oscillation.

The Hamiltonian $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

$$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2.$$

Factorizing the Hamiltonian

$$\hat{H} = \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right)$$

$$a^2 + b^2 = (a - ib)(a + ib)$$

$$\begin{aligned}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}) \\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}[\hat{x},\, \hat{p}] \\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\end{aligned}$$

Factorizing the Hamiltonian

We now define $\hat{A} = \hat{x} + \frac{i\hat{p}}{m\omega}$

and its Hermitian conjugate $\hat{A}^\dagger = \hat{x} - \frac{i\hat{p}}{m\omega}$

We therefore have $\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} = \hat{A}^\dagger\hat{A} + \frac{\hbar}{m\omega}$

Therefore, the Hamiltonian $\boxed{\hat{H} = \frac{1}{2}m\omega^2\hat{A}^\dagger\hat{A} + \frac{1}{2}\hbar\omega.}$

The commutator of $\hat{A}$ and $\hat{A}^\dagger$

$$[\hat{A},\ \hat{A}^\dagger] = \left[\hat{x} + \frac{i\hat{p}}{m\omega},\ \hat{x} - \frac{i\hat{p}}{m\omega}\right]$$

$$= -\frac{i}{m\omega}[\hat{x},\ \hat{p}] + \frac{i}{m\omega}[\hat{p},\ \hat{x}]$$

$$= \frac{2\hbar}{m\omega}$$

$$\left[\sqrt{\frac{m\omega}{2\hbar}}\hat{A},\ \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^\dagger\right] = 1$$

Creation and annihilation operators

Annihilation operator $\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A},$

Creation operator $\hat{a}^{\dagger} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^{\dagger}.$

$$\boxed{[\hat{a},\ \hat{a}^{\dagger}] = 1}$$

Creation and annihilation operators

Annihilation operator $$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A},$$

Creation operator $$\hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{A}^\dagger.$$

$$[\hat{a},\ \hat{a}^\dagger] = 1$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}}\,(\hat{a} - \hat{a}^\dagger)$$

Factorized Hamiltonian of SHO

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$= \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$E = \hbar\omega\left(N + \frac{1}{2}\right).$$

$$\hat{N} \equiv \hat{a}^\dagger\hat{a}.$$

Number operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

The ground state

On any normalized state ψ

$$\langle \hat{H} \rangle_\psi = (\psi, \hat{H}\psi) = \hbar\omega(\psi, \hat{a}^\dagger \hat{a}\psi) + \frac{1}{2}\hbar\omega(\psi, \psi)$$

$$= \hbar\omega(\hat{a}\psi, \hat{a}\psi) + \frac{1}{2}\hbar\omega \geq \frac{1}{2}\hbar\omega$$

$$(\psi, \psi) \geq 0$$

$$\hat{H}\psi = E\psi$$

$$E \geq \frac{1}{2}\hbar\omega$$

The ground state wave function

$$\hat{a}\psi_0 = 0$$

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\psi_0(x) = 0$$

$$\left(\hat{x} + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_0(x) = 0$$

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}x} = -\frac{m\omega}{\hbar}x\psi_0$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}}$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

Hamiltonian operator and number operator

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\hat{N} \equiv \hat{a}^\dagger\hat{a}.$$

$$\hat{a}\psi_0 = 0$$

$$\hat{N}\psi_0 = \hat{a}^\dagger\hat{a}\psi_0 = 0.$$

Thus ψ_0 is an eigenstate of the operator $\hat{N}$ with an eigenvalue $N = 0$. Therefore ψ_0 is an energy eigenstate with energy E_0 given by

$$E_0 = \hbar\omega\left(0 + \frac{1}{2}\right) = \frac{1}{2}\hbar\omega$$

Quantum Harmonic Oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$E_0 = \frac{1}{2}\hbar\omega$$

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