# Simple Harmonic Oscillator

Exited States | Orthonormality of states

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# The Hamiltonian of SHO

The total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

Here $\omega = \sqrt{k/m}$ is angular frequency of the oscillation.

The Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$

$$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2.$$

# Factorized Hamiltonian of SHO

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$\hat{N} \equiv \hat{a}^{\dagger}\hat{a}.$$

Number operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

# The ground state wave function and energy

$$\boxed{\hat{a}\psi_0 = 0}$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}}$$

$$\left(\hat{x} + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_0(x) = 0$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega}$$

# Operator manipulation

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)$$

$$\hat{N} \equiv \hat{a}^{\dagger}\hat{a}.$$

$$\hat{a}\psi_0 = 0$$

$$\hat{N}\psi_0 = \hat{a}^{\dagger}\hat{a}\psi_0 = 0.$$

Thus $\psi_0$ is an eigenstate of the operator $\hat{N}$ with an eigenvalue $N = 0$. Therefore $\psi_0$ is an energy eigenstate with energy $E_0$ given by

$$E_0 = \hbar\omega\left(0 + \frac{1}{2}\right) = \frac{1}{2}\hbar\omega$$

# Operator manipulation

$$[\hat{N},\, \hat{a}] = [\hat{a}^\dagger \hat{a},\, \hat{a}] = \hat{a}^\dagger [\hat{a},\, \hat{a}] + [\hat{a}^\dagger,\, \hat{a}]\hat{a} = -\hat{a}$$

$$[\hat{N},\, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a},\, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a},\, \hat{a}^\dagger] + [\hat{a}^\dagger,\, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger$$

Similarly we can show

$$[\hat{N},\, \hat{a}^2] = [\hat{N},\, \hat{a}\hat{a}] = [\hat{N},\, \hat{a}]\hat{a} + \hat{a}[\hat{N},\, \hat{a}] = -\hat{a}\hat{a} + \hat{a}(-\hat{a}) = -2\hat{a}^2,$$

$$[\hat{N},\, (\hat{a}^\dagger)^2] = [\hat{N},\, \hat{a}^\dagger \hat{a}^\dagger] = [\hat{N},\, \hat{a}^\dagger]\hat{a}^\dagger + \hat{a}^\dagger[\hat{N},\, \hat{a}^\dagger] = \hat{a}^\dagger \hat{a}^\dagger + \hat{a}^\dagger \hat{a}^\dagger = 2(\hat{a}^\dagger)^2$$

# Operator manipulation

$$[\hat{N},\, (\hat{a})^k] = -k(\hat{a})^k$$

$$[\hat{N},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^k$$

$$[\hat{a}^\dagger,\, (\hat{a})^k] = -k(\hat{a})^{k-1}$$

$$[\hat{a},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^{k-1}$$

$$\text{If } \hat{A}\psi = 0, \text{ then } \hat{A}\hat{B}\psi = [\hat{A},\, \hat{B}]\psi$$

# Fist excited states

Since $\hat{a}$ annihilates $\psi_0$ consider acting on the ground state with $\hat{a}^\dagger$. It is clear that $\hat{a}^\dagger$ cannot also annihilate $\psi_0$. If that would happen then acting with both sides of the commutator identity $[\hat{a},\, \hat{a}^\dagger] = 1$ on $\psi_0$ would lead to a contradiction: the left-hand side would vanish but the right-hand side would not. Thus consider the wave function

$$\boxed{\psi_1 \equiv \hat{a}^\dagger \psi_0}$$

We are going to show that this is an energy eigenstate.

# Fist excited states

$$\hat{N}\psi_1 = \hat{N}\hat{a}^\dagger\psi_0 = [\hat{N},\, \hat{a}^\dagger]\psi_0 = \hat{a}^\dagger\psi_0 = \psi_1$$

- $\psi_0$ is an eigenstate of the operator $\hat{N}$ with eigenvalue $N = 0$.
- $\psi_1$ is an eigenstate of the operator $\hat{N}$ with eigenvalue $N = 1$.
- $\hat{a}^\dagger$ acting on $\psi_0$ increases the eigenvalue of $\hat{N}$ by one unit.
- $\hat{a}^\dagger$ is called the *creation* operator or the *raising* operator.

# Fist excited states

Moreover

$$(\psi_1,\, \psi_1) = (\hat{a}^\dagger \psi_0,\, \hat{a}^\dagger \psi_0) = (\psi_0,\, \hat{a}\hat{a}^\dagger \psi_0)$$

$$= (\psi_0,\, [\hat{a},\, \hat{a}^\dagger]\psi_0) = (\psi_0,\, \psi_0) = 1$$

$\psi_1$ is normalized and is the wave function of first exited state.

Energy of the first exited state $$E_1 = \hbar\omega\left(1 + \frac{1}{2}\right) = \frac{3}{2}\hbar\omega$$

# Second Excited states

Next consider the state $\psi_2' \equiv \hat{a}^\dagger \hat{a}^\dagger \psi_0.$

$$\hat{N}\psi_2' = \hat{N}\hat{a}^\dagger\hat{a}^\dagger\psi_0 = [\hat{N},\, \hat{a}^\dagger\hat{a}^\dagger]\psi_0 = 2\hat{a}^\dagger\hat{a}^\dagger\psi_0 = 2\psi_2'$$

$\psi_2'$ is a state with number $N = 2$ and energy $E_2 = \frac{5}{2}\hbar\omega.$

$$\begin{aligned}(\psi_2',\, \psi_2') &= (\hat{a}^\dagger\hat{a}^\dagger\psi_0,\, \hat{a}^\dagger\hat{a}^\dagger\psi_0) = (\psi_0,\, \hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\psi_0) = (\psi_0,\, \hat{a}[\hat{a},\, \hat{a}^\dagger\hat{a}^\dagger]\psi_0) \\ &= (\psi_0,\, 2\hat{a}\hat{a}^\dagger\psi_0) = 2(\psi_0,\, \psi_0) = 2.\end{aligned}$$

# Excited state wave functions

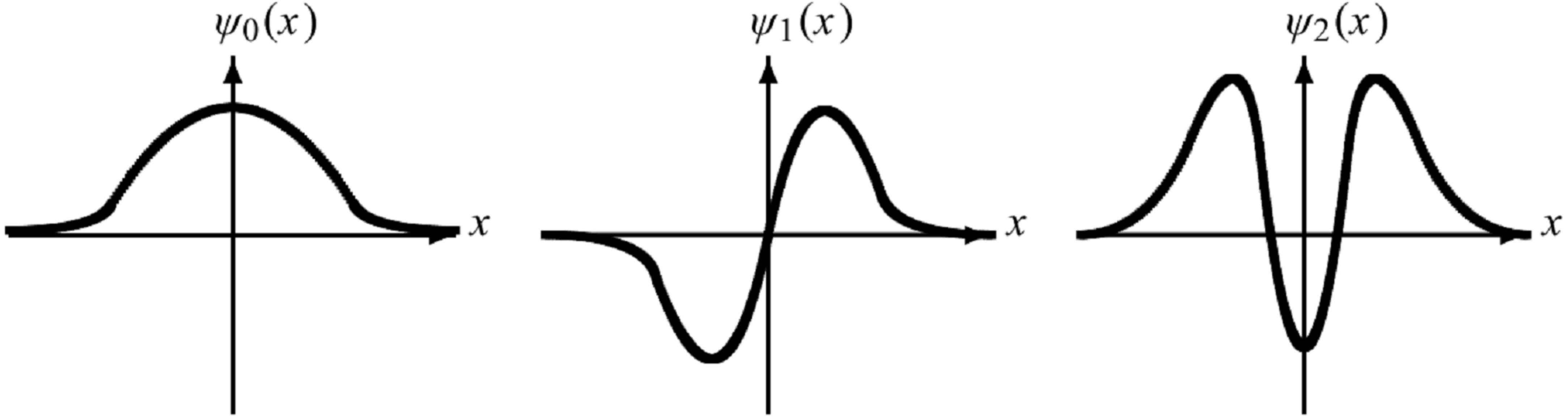
The properly normalized wave function is

$$\psi_2 \equiv \frac{1}{\sqrt{2}}\hat{a}^\dagger\hat{a}^\dagger\psi_0.$$

$$\psi_3 \equiv \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}}\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger\psi_0 = \frac{1}{\sqrt{3!}}(\hat{a}^\dagger)^3\psi_0$$

$$\psi_n \equiv \frac{1}{\sqrt{n!}}\underbrace{\hat{a}^\dagger\cdots\hat{a}^\dagger}\psi_0 = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0.$$

# Shapes of the first three wave functions of the SHO




$$\psi_n \equiv \frac{1}{\sqrt{n!}} \underbrace{\hat{a}^\dagger \cdots \hat{a}^\dagger}\psi_0 = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \psi_0.$$

# Eigenvalue of $\psi_n$ for the operators $\widehat{N}$ and $\widehat{H}$

$$\begin{aligned}
\hat{N}\psi_n &= \frac{1}{\sqrt{n!}}\hat{N}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}\hat{a}^\dagger\hat{a}\,(\hat{a}^\dagger)^n\psi_0 \\
&= \frac{1}{\sqrt{n!}}\hat{a}^\dagger[\hat{a},\,(\hat{a}^\dagger)^n]\psi_0 = \frac{1}{\sqrt{n!}}\hat{a}^\dagger\,n(\hat{a}^\dagger)^{n-1}\psi_0 \\
&= \frac{n}{\sqrt{n!}}\,(\hat{a}^\dagger)^n\psi_0 = n\,\psi_n
\end{aligned}$$

Since for the operator $\hat{N}$ the eigenvalue of $\psi_n$ is $n$, the energy eigenvalue $E_n$ is given be

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

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