Simple Harmonic Oscillator

Expectation values of $\hat{x}$, $\hat{p}$, $\hat{x}^2$ and $\hat{p}^2$ | Uncertainty principle

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The Hamiltonian of SHO

The total energy $$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

Here $\omega = \sqrt{k/m}$ is angular frequency of the oscillation.

The Hamiltonian $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

$$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2.$$

Factorized Hamiltonian of SHO

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$\hat{N} \equiv \hat{a}^\dagger\hat{a}.$$

Number operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

Wave functions and energy eigenvalues

$$\psi_n \equiv \frac{1}{\sqrt{n!}} \underbrace{\hat{a}^\dagger \cdots \hat{a}^\dagger}\psi_0 = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n \psi_0$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$E_0 = \frac{1}{2}\hbar\omega$$

Operator manipulation

$$[\hat{N},\, (\hat{a})^k] = -k(\hat{a})^k$$

$$[\hat{N},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^k$$

$$[\hat{a}^\dagger,\, (\hat{a})^k] = -k(\hat{a})^{k-1}$$

$$[\hat{a},\, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^{k-1}$$

$$\text{If } \hat{A}\psi = 0, \text{ then } \hat{A}\hat{B}\psi = [\hat{A},\, \hat{B}]\psi$$

Properties of of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{a}\psi_n = \hat{a}\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}[\hat{a},\, (\hat{a}^\dagger)^n]\psi_0 = \frac{n}{\sqrt{n!}}(\hat{a}^\dagger)^{n-1}\psi_0$$

$$= \frac{n}{\sqrt{n!}}\sqrt{(n-1)!}\,\psi_{n-1} = \sqrt{n}\,\psi_{n-1}$$

$$\begin{aligned}\hat{a}^\dagger\psi_n &= \hat{a}^\dagger\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0 = \frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^{n+1}\psi_0 \\ &= \frac{1}{\sqrt{n!}}\sqrt{(n+1)!}\,\psi_{n+1} = \sqrt{(n+1)}\,\psi_{n+1}\end{aligned}$$

Orthonormality of eigenstates

Eigenstates are orthonormal: $$(\psi_m, \psi_n) = \delta_{mn}$$

$$\begin{aligned} \hat{a}\psi_n &= \sqrt{n}\,\psi_{n-1}, \\ \hat{a}^\dagger\psi_n &= \sqrt{(n+1)}\,\psi_{n+1}. \end{aligned}$$

Kronecker delta

$$\delta_{mn} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Expectation values of operators

$$
\begin{aligned}
\langle \hat{x} \rangle_{\psi_n} &= (\psi_n,\, \hat{x}\psi_n) = \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, (\hat{a} + \hat{a}^\dagger)\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}\psi_n + \hat{a}^\dagger\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}\psi_n) + \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \hat{a}^\dagger\psi_n) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \sqrt{n}\,\psi_{n-1}) + \sqrt{\frac{\hbar}{2m\omega}}\,(\psi_n,\, \sqrt{(n+1)}\,\psi_{n+1}) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\,\sqrt{n}(\psi_n,\, \psi_{n-1}) + \sqrt{\frac{\hbar}{2m\omega}}\,\sqrt{(n+1)}(\psi_n,\, \psi_{n+1}) \\
&= 0.
\end{aligned}
$$

Expectation values of operators

$$
\begin{aligned}
\langle \hat{x}^2 \rangle_{\psi_n} &= (\psi_n,\, \hat{x}^2 \psi_n) = \frac{\hbar}{2m\omega} (\psi_n,\, (\hat{a} + \hat{a}^\dagger)^2 \psi_n) \\
&= \frac{\hbar}{2m\omega} (\psi_n,\, (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) \psi_n) \\
&= \frac{\hbar}{2m\omega} (\psi_n,\, (\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) \psi_n) \\
&= \frac{\hbar}{2m\omega} (\psi_n,\, (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \psi_n) \\
&= \frac{\hbar}{2m\omega} (\psi_n,\, (1 + 2\hat{N}) \psi_n) \\
&= \frac{\hbar}{2m\omega} (1 + 2n)
\end{aligned}
$$

$$
\hat{a}\hat{a}^\dagger = [\hat{a},\, \hat{a}^\dagger] + \hat{a}^\dagger\hat{a} = 1 + \hat{N}
$$

Expectation values of operators

$$\langle \hat{p} \rangle_{\psi_n} = 0$$

$$\langle \hat{p}^2 \rangle_{\psi_n} = \frac{m\hbar\omega}{2}(1+2n).$$

$$\langle \hat{x} \rangle_{\psi_n} = 0$$

$$\langle \hat{x}^2 \rangle_{\psi_n} = \frac{\hbar}{2m\omega}(1+2n)$$

$$\frac{m\omega^2}{2}\langle \hat{x}^2 \rangle_{\psi_n} = \frac{1}{2m}\langle \hat{p}^2 \rangle_{\psi_n} = \frac{1}{2}\langle \hat{H} \rangle_{\psi_n}$$

Uncertainty principle

$$\langle \hat{x} \rangle_{\psi_n} = 0$$

$$\langle \hat{x}^2 \rangle_{\psi_n} = \frac{\hbar}{2m\omega}(1+2n)$$

$$\langle \hat{p} \rangle_{\psi_n} = 0$$

$$\langle \hat{p}^2 \rangle_{\psi_n} = \frac{m\hbar\omega}{2}(1+2n).$$

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$

$$\Delta p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$$

$$\Delta x \Delta p = \left(n + \frac{1}{2}\right)\hbar$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

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