Quantum Mechanics

Angular Momentum

Operators | Hermiticity | Commutation

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Angular momentum

In classical physics the angular momentum of a particle with momentum **p** and position **r** is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Hence the components of $\mathbf{L} = (L_x, L_y, L_z)$ are given by

$$L_x = yp_z - zp_y,$$
$$L_y = zp_x - xp_z,$$
$$L_z = xp_y - yp_x.$$

Angular momentum operator

The angular momentum operator $\hat{\mathbf{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ can be obtained by replacing $\mathbf{r}$ and $\mathbf{p}$ by the corresponding operators in the position representation:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Hermiticity of angular momentum operator

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$$

$$\begin{aligned}
(\hat{L}_x)^\dagger &= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)^\dagger \\
&= (\hat{y}\hat{p}_z)^\dagger - (\hat{z}\hat{p}_y)^\dagger \\
&= (\hat{p}_z)^\dagger(\hat{y})^\dagger - (\hat{p}_y)^\dagger(\hat{z})^\dagger \\
&= \hat{p}_z\hat{y} - \hat{p}_y\hat{z} \\
&= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\
&= \hat{L}_x
\end{aligned}$$

$$\hat{L}_x^\dagger = \hat{L}_x, \qquad \hat{L}_y^\dagger = \hat{L}_y, \qquad \hat{L}_z^\dagger = \hat{L}_z.$$

Commutation relations

$$
\begin{aligned}
[\hat{L}_x,\ \hat{L}_y] &= [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y,\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z,\ \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z,\ \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y,\ \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y,\ \hat{x}\hat{p}_z] \\
&= [\hat{y}\hat{p}_z,\ \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y,\ \hat{x}\hat{p}_z] \\
&= \hat{y}[\hat{p}_z,\ \hat{z}]\hat{p}_x + \hat{x}[\hat{z},\ \hat{p}_z]\hat{p}_y \\
&= \hat{y}(-i\hbar)\hat{p}_x + \hat{x}(i\hbar)\hat{p}_y \\
&= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x) \\
&= i\hbar\hat{L}_z.
\end{aligned}
$$

Commutation relations

Orbital angular momentum

$$[\hat{L}_x,\, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y,\, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z,\, \hat{L}_x] = i\hbar\hat{L}_y.$$

Spin angular momentum

$$[\hat{S}_x,\, \hat{S}_y] = i\hbar\hat{S}_z, \qquad [\hat{S}_y,\, \hat{S}_z] = i\hbar\hat{S}_x, \qquad [\hat{S}_z,\, \hat{S}_x] = i\hbar\hat{S}_y.$$

Simultaneous eigenstates of angular momentum

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y.$$

$$\hat{L}_x\psi_0 = \lambda_x\psi_0$$
$$\hat{L}_y\psi_0 = \lambda_y\psi_0$$
$$\hat{L}_z\psi_0 = \lambda_z\psi_0$$

$$\begin{aligned} i\hbar\hat{L}_z\psi_0 &= [\hat{L}_x, \hat{L}_y]\psi_0 \\ &= \hat{L}_x\hat{L}_y\psi_0 - \hat{L}_y\hat{L}_x\psi_0 \\ &= \hat{L}_x\lambda_y\psi_0 - \hat{L}_y\lambda_x\psi_0 \\ &= (\lambda_x\lambda_y - \lambda_y\lambda_x)\psi_0 \\ &= 0 \end{aligned}$$

$$\hat{L}_x\psi_0 = \hat{L}_y\psi_0 = \hat{L}_z\psi_0 = 0.$$

$$\lambda_z = 0.$$

Simultaneous eigenstates of angular momentum

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\begin{aligned}
[\hat{L}_z,\, \hat{\mathbf{L}}^2] &= [\hat{L}_z,\, \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z] \\
&= [\hat{L}_z,\, \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y] \\
&= [\hat{L}_z,\, \hat{L}_x]\hat{L}_x + \hat{L}_x[\hat{L}_z,\, \hat{L}_x] + [\hat{L}_z,\, \hat{L}_y]\hat{L}_y + \hat{L}_y[\hat{L}_z,\, \hat{L}_y] \\
&= i\hbar\hat{L}_y\hat{L}_x + i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_x\hat{L}_y - i\hbar\hat{L}_y\hat{L}_x \\
&= 0.
\end{aligned}$$

Angular momentum operator

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Thank you for watching

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