Quantum Mechanics

Angular Momentum

Eigenvalues | Eigenfunctions | Spherical harmonics

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Angular momentum operator

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Commutation relations

Orbital angular momentum

$$[\hat{L}_x,\, \hat{L}_y] = i\hbar\hat{L}_z, \qquad [\hat{L}_y,\, \hat{L}_z] = i\hbar\hat{L}_x, \qquad [\hat{L}_z,\, \hat{L}_x] = i\hbar\hat{L}_y.$$

$$\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Spin angular momentum

$$[\hat{S}_x,\, \hat{S}_y] = i\hbar\hat{S}_z, \qquad [\hat{S}_y,\, \hat{S}_z] = i\hbar\hat{S}_x, \qquad [\hat{S}_z,\, \hat{S}_x] = i\hbar\hat{S}_y.$$

Angular momentum in spherical coordinates

$$x = r\sin\theta\cos\phi, \qquad r = \sqrt{x^2 + y^2 + z^2},$$

$$y = r\sin\theta\sin\phi, \qquad \theta = \cos^{-1}\left(\frac{z}{r}\right),$$

$$z = r\cos\theta, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Angular momentum in spherical coordinates

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0$$

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial \phi}}$$

Angular momentum in spherical coordinates

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right),$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}.$$

$$\boxed{\hat{\mathbf{L}}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].}$$

Eigenvalues of angular momentum

$$\hat{L}_z\,\psi_{lm}(\theta,\phi) = \hbar m\,\psi_{lm}(\theta,\phi), \qquad m \in \mathbb{R}$$

$$\hat{\mathbf{L}}^2\,\psi_{lm}(\theta,\phi) = \hbar^2\,l(l+1)\,\psi_{lm}(\theta,\phi), \qquad l \in \mathbb{R}.$$

$$\begin{aligned}(\psi,\,\hat{\mathbf{L}}^2\psi) &= (\psi,\,\hat{L}_x^2\psi) + (\psi,\,\hat{L}_y^2\psi) + (\psi,\,\hat{L}_z^2\psi)\\ &= (\hat{L}_x\psi,\,\hat{L}_x\psi) + (\hat{L}_y\psi,\,\hat{L}_y\psi) + (\hat{L}_z\psi,\,\hat{L}_z\psi) \geq 0\end{aligned}$$

Eigenvalues of angular momentum

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$-i\hbar\frac{\partial\psi_{lm}}{\partial\phi} = \hbar m\,\psi_{lm}$$

$$\frac{\partial\psi_{lm}}{\partial\phi} = im\,\psi_{lm}$$

$$\psi_{lm}(\theta,\phi) = e^{im\phi}\,P_l^m(\theta)$$

$$\psi_{lm}(\theta,\phi+2\pi) = \psi_{lm}(\theta,\phi).$$

$$e^{im(\phi+2\pi)} = e^{im\phi}$$

$$e^{i2\pi m} = 1$$

$$m \in \mathbb{Z}$$

Eigenvalues of angular momentum

$$\hat{L}_z \, \psi_{lm}(\theta, \phi) = \hbar m \, \psi_{lm}(\theta, \phi),$$

$$\hat{\mathbf{L}}^2 \, \psi_{lm}(\theta, \phi) = \hbar^2 \, l(l+1) \, \psi_{lm}(\theta, \phi),$$

$$\boxed{l = 0,\ 1,\ 2,\ 3,\ \ldots}$$

$$\boxed{m \in \mathbb{Z}}$$

$$\boxed{-\,l \le m \le l.}$$

Eigenfunctions of angular momentum

Spherical harmonics

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}, \qquad (m \geq 0).$$

Here $P_l^m(\cos\theta)$ is the associated Legendre functions. $m < 0$, we use

$$Y_{l,m}(\theta,\phi) = (-1)^m [Y_{l,-m}(\theta,\phi)]^*.$$

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, \sin\theta\, Y^*_{l'm'}(\theta,\phi)\, Y_{lm}(\theta,\phi) = (Y^*_{l'm'},\, Y_{lm}) = \delta_{l'l}\,\delta_{m'm}.$$

Eigenfunctions of angular momentum

Spherical harmonics

$$Y_{lm}(\theta,\phi) = (-1)^m \sqrt{\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}, \qquad (m \geq 0).$$

$$\hat{L}_z Y_{lm} = \hbar m Y_{lm},$$

$$\hat{\mathbf{L}}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm},$$

$$m \in \mathbb{Z}$$

$$l = 0,\ 1,\ 2,\ 3,\ \ldots$$

$$-l \leq m \leq l.$$

Eigenfunctions of angular momentum

$Y_{lm}(\theta, \varphi)$	$Y_{lm}(x, y, z)$
$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{10}(x, y, z) = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
$Y_{1,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$	$Y_{1,\pm1}(x, y, z) = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$
$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$	$Y_{20}(x, y, z) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$
$Y_{2,\pm1}(\theta, \varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta\cos\theta$	$Y_{2,\pm1}(x, y, z) = \mp\sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2}$
$Y_{2,\pm2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$	$Y_{2,\pm2}(x, y, z) = \sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$

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